Ambiguity Averse Portfolio Choices in an Aging Population*

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Abstract

This paper studies how wealth and aging affect portfolio choices in a life-cycle model with ambiguity aversion. In the absence of ambiguity, the model delivers a constant share of wealth invested in risky assets for all agents, as in Merton (1969). With ambiguity aversion, wealthier and older agents become endogenously more optimistic about risky asset returns, relative to poorer younger agents. This delivers an age profile of risky asset shares in line with that in US data. As life expectancy grows, old agents become even more optimistic, while young agents become more pessimistic, amplifying the age gaps in portfolio composition, and implying further increases in intergenerational inequality.
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1 Introduction

When faced with Knightian uncertainty, ambiguity averse agents over-weight the probability of the 'worst case' outcomes with low utility realizations (Gilboa and Schmeidler, 1989; Hansen and Sargent, 2008). These belief distortions differ according to the agent's environment, preferences, and characteristics: the worst case is very different, for example, for wealthy and poor households (Michelacci and Paciello, 2020, 2023), or between different policy regimes (Ferriere and Karantounias, 2019). As a consequence, when an economy goes through a structural shift, the belief distortions due to ambiguity aversion may change, with implications for a range of aggregate and distributional outcomes.

In this paper, we explore these effects for one of the key shifts taking place in developed economies today: population aging. In particular, we ask how increased lifespans affect ambiguity-averse beliefs about asset returns, and what that implies for portfolio choices and asset prices. This is likely to be important for policy in the coming decades, as savings behavior has been at the forefront of recent discussions of economic policy under demographic change (Goodhart and Pradhan, 2020; Auclert et al., 2021; Kopecky and Taylor, 2022). Moreover, there is substantial empirical evidence that ambiguity aversion is a key determinant of portfolio choices and asset prices (Antoniou et al., 2015; Dimmock et al., 2016; Collard et al., 2018; Corgnet et al., 2020).

To analyse the interaction between aging, portfolio choice, and ambiguity aversion, we build an overlapping-generations model with ambiguity over risky asset returns. We find that increases in life expectancy cause young agents to distort their beliefs more strongly towards pessimistic outcomes, while older agents in contrast become more optimistic. Population aging therefore increases the concentration of equity holdings among older households, as their relatively more optimistic beliefs drive them to choose higher risky asset shares than the young. This in turn implies older generations become relatively wealthier, as they earn greater asset returns than younger agents. In a quantitative illustration, the model predicts that plausible longevity increases over the next 80 years will cause the age-profile of risky asset shares to steepen by 9%. In addition, this population aging also substantially increases the heterogeneity in portfolio holdings within cohorts.

As in Eggertsson et al. (2019) and Malmendier et al. (2020), we mostly focus on a simple case of the model with maximum three-period lifespans. This allows us to obtain analytic results, and inspect the mechanisms at work. We model ambiguity using so-called 'multiplier preferences', in which agents whose payoffs are more exposed to ambiguity distort their beliefs more strongly towards low-utility states (Hansen et al., 1999). Although all investors view low returns as the worst case, a given drop in returns is much more damaging to some investors than others, and those who would be particularly badly affected do more to insure themselves against that possibility.[1]

We begin our analysis in a small open economy aging alone, in which safe asset returns and the distribution of risky asset returns are held fixed. We find that ambiguity aversion endogenously generates return expectations which are increasing in wealth and age. This is consistent with survey evidence on return expectations (Giglio et al., 2021), experiments on biases in financial decision-making (Kovalchik et al., 2005), and the observation that young and poor households typically hold less risky portfolios than those further up the age and wealth distribution (Chang et al., 2018; Catherine, 2022). Note that we match this last result despite the fact that, in the absence of ambiguity, the model reduces to a simple Merton (1969)-style model in which risky asset shares are constant for all agents.

Importantly, the effects of age on responses to ambiguity are driven by changes in *life expectancy*, rather than the number of years an agent has lived. As a consequence, reductions in mortality rates for older agents cause changes in the age profile of risky asset shares. Specifically, as mortality rates fall young and old agents adjust their return expectations in opposite directions. The young become more pessimistic, distorting beliefs more strongly towards low risky asset returns, and investing less as a result - while simultaneously older agents become more optimistic, and invest more. As populations age around the world, this has consequences for the future of intergenerational inequality and the composition of asset demand.

These endogenous changes in responses to ambiguity with wealth and life expectancy come about because of the interaction of two distinct channels. The first is the 'wealth channel', in which an agent who is saving more is more exposed in monetary terms to low asset returns. They have more 'skin in the game', and so have a greater desire to make their decisions robust to returns ambiguity, implying more pessimistic expectations.[2] The second is the 'marginal utility channel', in which an agent who expects to have a high marginal utility of consumption in the next period suffers a greater utility loss from a given fall in wealth. A poor agent may not lose much in monetary terms from a fall in returns, but due to the curvature of standard utility functions, they have a high marginal utility of consumption, and so of wealth. A small monetary loss can therefore have large utility consequences for these agents.

As life expectancy increases, younger agents save more, to fund consumption in their now-extended old age. They also expect to do the same throughout their middle age, implying lower consumption in the immediate future. With a diminishing marginal utility

[1]Several existing models of ambiguity aversion in portfolio choice instead assume all investors distort beliefs to the lowest return among a given set of possibilities, the range of which is typically specified as an exogenous aspect of preferences. This distinction is discussed further in the Related Literature section below.

[2]This echoes the literature arguing wealthier households have more incentive to process information about asset returns, for the same reason (Arrow, 1987; Lei, 2019; Macaulay, 2021).

of consumption, this means that through both channels they become more exposed to shortfalls in asset returns, so they distort their beliefs more towards low returns to make their decisions more robust to ambiguity. As a result they invest less in risky assets, and instead allocate more of their savings to the risk-free asset.

For agents in middle age, the wealth channel operates in the same way. However, the marginal utility channel is reversed. When the probability of surviving into old age is low, they do not save much for those potential future periods. Conditional on survival, their old-age consumption is therefore low. As the mortality rate falls and life expectancy rises, they save more for their old age, which implies greater consumption in that period. As such, rising life expectancy reduces the expected marginal utility of consumption in old age, implying utility is *less* exposed to a given fall in wealth. Through the marginal utility channel, middle-aged agents therefore become more optimistic about asset returns, and increase their portfolio share in risky assets.

Which of these channels dominates is regulated by a simple condition: with CRRA preferences, the marginal utility channel dominates if and only if the elasticity of intertemporal substitution (EIS) is less than 1. In that case marginal utility changes sharply with (future) wealth, and so outweighs the wealth channel. Intuitively, this is related to the standard result that income effects of interest rate changes dominate substitution effects when the EIS is less than 1 (see Flynn et al., 2023, for an extended discussion). Substitution effects are small when marginal utility is highly convex, and this is precisely when our marginal utility channel is large. Attempts to measure the EIS among households typically find values below 1 (Havránek, 2015), so we take this as our baseline.

A similar logic drives the effects of wealth on return expectations. An increase in wealth implies agents save more, making them more exposed to return fluctuations. At the same time, they expect to be wealthier in the future, which reduces their expected marginal utility of wealth. As with changes in life expectancy, the marginal utility channel dominates whenever the EIS is less than 1, in which case wealth reduces the extent to which agents distort return beliefs due to ambiguity aversion. Interestingly, this can be seen as an alternative explanation for the result in the literature on experience-based-learning that households who experience high returns are more optimistic about returns in the future (Malmendier and Nagel, 2011; Foltyn, 2020).

After characterizing these channels, we extend the model to a closed economy, in which the equity premium is endogenous. Initially, as life expectancy rises from a low level, the dominant force is the increasing optimism of the middle-aged agents. Aggregate demand for risk rises, and the equity premium falls. However, as life expectancy continues to grow, increases in middle-aged optimism slow down, and are eventually dominated by the greater pessimism of the young. Past a certain threshold, aggregate demand for risky assets begins to decline, and the equity premium rises as a result. This occurs because the effects of age on beliefs are smaller in the model for agents with more wealth. As mortality rates fall, young agents save more for their old age, middle-aged agents become wealthier, and so middle-aged agents become increasingly unresponsive to further increases in life expectancy. Interestingly, although the model is very stylized, this result is consistent with the U-shaped evolution of the equity premium observed across developed economies since 1950 (Kuvshinov and Zimmermann, 2020).

Finally, we end with an illustration of how these effects might play out with plausible degrees of demographic change in the coming decades. We extend the model to 100-year-plus maximum lifespans, and calibrate it to data on wealth and mortality rates in the US in 2019. The model produces risky portfolio shares that are increasing and concave in wealth, as in data from the Survey of Consumer Finances. We then simulate the effect of an increase in life expectancy, replacing the calibrated 2019 mortality rates with projected mortality rates for 2100. As in the analytical model, young agents become more pessimistic about asset returns relative to older agents, so the gap between the risky asset shares of old and young widens. Comparing those with median levels of wealth in each cohort, the increase in life expectancy increases the slope of the age-profile of risky asset shares by 9%. Going further, we also find that the effects of aging on beliefs are substantially more pronounced among those with lower levels of wealth. Overall, this means that the simulated aging to the year 2100 leads to large increases in within-generation portfolio heterogeneity, on top of the effects across generations.

Related Literature: We principally contribute to the literatures on ambiguity aversion, demographic change, and life-cycle portfolio choice in macroeconomics and finance.

Ambiguity aversion has been successful in providing theoretical explanations for a number of phenomena in macroeconomics and finance (see reviews in Ilut and Schneider (2022) and Epstein and Schneider (2010)). Our work is particularly related to models in which agents can endogenously adjust their response to ambiguity based on their own exposure to the variable(s) in question (Hansen et al., 1999; Cagetti et al., 2002; Bidder and Smith, 2012),[3] even with constant preferences. In particular, Michelacci and Paciello (2023, 2020) show that with ambiguity aversion wealthy and poor households hold systematically different expectations of aggregate variables. This explains several features of survey data on expectations, and influences macroeconomic dynamics. Our

[3]These examples, like us, use so-called 'multiplier preferences', in which agents distort beliefs towards a worst-case model, subject to a penalty of belief distortions which is linear in the relative entropy between the central and worst-case model used. An alternative approach that also allows endogenous adjustment of belief distortions is the smooth ambiguity preferences of Klibanoff et al. (2005, 2009), adapted to macroeconomic settings in Altug et al. (2020). Chen et al. (2014) use this in a model of portfolio choice, but do not study changes over the life-cycle or with age.

model extends these insights to portfolio choice, and shows that demographic changes therefore affect inequality and the equity premium.

Several of our results for how the belief distortions driven by ambiguity change with age and wealth depend on whether the elasticity of intertemporal substitution is greater or less than 1. In this we therefore add to the insights of Ferriere and Karantounias (2019) and Balter et al. (2022), who show that the same condition determines the outcome of models of optimal fiscal policy and momentum in asset return expectations respectively, once ambiguity aversion is present. Recent empirical evidence has tended to favor an EIS substantially below 1 (Havránek, 2015; Crump et al., 2022), so we take this as the baseline case for our analysis.

Our results are relevant for the literature on how demographic change will affect asset markets and inequality. A variety of papers have studied demographic effects on aggregate asset demand by holding age profiles of asset holdings or savings rates fixed, and changing the proportions of households within each age group in line with past demographic data, or future projections (e.g. Mankiw and Weil, 1989; Mian et al., 2021).[4] This approach only yields sufficient statistics for aggregate asset demand if household decision rules depend on that household's age, but are otherwise independent of aggregate demographic change (Auclert et al., 2021). We show that under ambiguity aversion, that is not the case, as changes in life expectancy affect these decision rules.

Finally, we also relate to the large literature on portfolio choice over the life cycle (see Gomes et al., 2021, for a review). Within this literature, a number of papers have proposed mechanisms to explain why older households typically invest the same or greater shares of their wealth in risky assets than young households. This pattern, while not present in standard life-cycle models (Cocco et al., 2005; Gomes, 2020), can be generated by declines in labor market uncertainty as households age (Chang et al., 2018), or the cyclicality of return skewness (Catherine, 2022). Indeed, like us, Campanale (2011) and Peijnenburg (2018) offer explanations of the data based on ambiguity aversion.

We view the mechanism in this paper as complementary to these other forces. The key conceptual distinction between us and the existing literature is that in many of these previous papers, portfolio choices depend explicitly on the number of years an agent has lived to date. In contrast, the endogenous responses to ambiguity in our model imply that return expectations and portfolio decisions depend on the number of years an agent *expects* to live in the future.[5] In Peijnenburg (2018), for example, savers face Knightian

[4] An alternative approach analyses quantitative life-cycle models with rational expectations (e.g. Carvalho et al., 2016; Kopecky and Taylor, 2022), which also abstract from the mechanisms we study.

[5] Note that this implies the same mechanisms operate for any increase in subjective survival probabilities, even if that is not reflected in objective reality (such deviations are studied in e.g. Grevenbrock et al. (2021)). We abstract from this in this model to highlight the effects through returns ambiguity.

uncertainty over a bounded interval of possible mean asset returns. With every period of life, they observe some returns data, and so are able to shrink that interval. Since ambiguity-averse agents in that model set return expectations to the lower bound of the plausible interval, that learning results in expected risky asset returns that rise with age.[6] In our model, we instead consider a constant preference for robustness, and abstract from reductions in ambiguity through learning.[7] In this environment, we show that older households are typically less vulnerable to return shortfalls, and so choose to react less to their ambiguity. This mechanism therefore implies ambiguity aversion can generate an upward-sloping age profile of risky asset shares, even if young poor households learn nothing about asset markets, as would be the case if they choose not to pay attention to them (as in e.g. Lei, 2019; Macaulay, 2021). Importantly, our mechanism also means that the age-profile of risky asset shares changes with life expectancy: as the population ages, older households become even more optimistic about asset returns relative to the young, and so the age gradient of risky asset shares gets steeper.

The rest of the paper is organized as follows. Section 2 sets out the model environment with a general maximum lifespan. Section 3 characterizes the effects of ambiguity aversion and demographic change analytically in the special case where the maximum lifespan is 3 periods. Section 4 calibrates the model to the US in 2019, and explores the implications for the age profile of risky asset shares both now and in the future. Section 5 concludes.

2 Model

2.1 Environment

Demographics: In each period t, a continuum of agents with measure 1 are born with age $j = 0$. An agent of age j survives to age $j + 1$ in the next period with exogenous probability ϕ_j. We set $\phi_J = 0$, implying a maximum age of J. There is no population growth.

Preferences: An agent of age j chooses consumption and their portfolio allocation to

[6] Similarly, in Chang et al. (2018) labor market uncertainty declines as households age, which enables them to take more risk in their asset portfolios.

[7] A related model with multiplier preferences is Maenhout (2004). However, in that model the preference for robustness is normalized by wealth. While this normalization makes the model more tractable, it also assumes away many of the changes in belief distortions we study, and so abstracts from the mechanisms in our model.

maximize expected discounted lifetime utility:

$$U_{j,t} = E_{j,t} \sum_{k=j}^{J} \left[\beta^{k-j} \Phi_{j,k} \frac{c_{k,t+k-j}^{1-\gamma}}{1-\gamma} \right] \tag{1}$$

where β is the discount factor, $c_{j,t}$ is the consumption of an agent with age j in period t, and γ is relative risk aversion. $\Phi_{j,k}$ is the cumulative probability of surviving to age k, conditional on having survived to age j, defined as:

$$\Phi_{j,k} = \begin{cases} 1 & \text{for } k = j \\ \prod_{x=j}^{k-1} \phi_x & \text{for } k > j \end{cases} \tag{2}$$

Ambiguity aversion affects choices because it distorts the expectations operator $E_{j,t}$ away from the expectations calculated under the objective probability distribution of future outcomes. Specifically, ambiguity averse agents over-weight the probabilities of future states with low utility, and underweight states with high utility (Gilboa and Schmeidler, 1989; Hansen and Sargent, 2008).

Endowment and Savings: Agents are born with an endowment of w_0 units of wealth. There is no labor income.[8] The wealth of an agent of age j in period t is denoted $w_{j,t}$.

There are two assets available for savings: a risk-free bond with gross interest rate R_f, and a risky asset with a gross return of R_t. The return on the risky asset is such that $\log(R_t)$ has an i.i.d. Normal distribution with mean $\tilde{\mu}$ and variance σ^2. For the results below, it will be helpful to define $\mu = \tilde{\mu} + \sigma^2/2$, where μ is the logarithm of $E_t R_{t+1}$.

Finally, we impose that $w_{J+1} \geq 0$, which prevents agents from borrowing when they know for certain they are in their final period. There are no bequests, so by assumption if an agent dies with positive asset holdings those assets are destroyed.

2.2 Value Functions

Denoting the share of the agent's period-t wealth invested in risky assets as $\alpha_{j,t} \in [0, 1]$,[9] the agent's utility maximization problem can be written as:

$$V_j(w_j) = \max_{c_{j,t}, \alpha_{j,t}} \left\{ \frac{c_{j,t}^{1-\gamma}}{1-\gamma} + \beta \phi_j E[V_{j+1}(w_{j+1})] \right\} \tag{3}$$

[8] The advantage of this assumption is that, in the absence of ambiguity aversion, it implies a constant portfolio share in risky assets for all ages (Merton, 1969). All changes in portfolio shares in our model must therefore come from the interaction of ambiguity aversion and aging.

[9] In all of the analysis here and in Section 3 we focus on the interior solution where these constraints on $\alpha_{j,t}$ do not bind. They will however become relevant in Section 4.

subject to:

$$w_{j+1} = (w_j - c_j) R^p_{j,t+1} \tag{4}$$

$$w_{J+1} \geq 0 \tag{5}$$

where $R^p_{j,t+1}$ is the agent's total return on their portfolio:

$$R^p_{j,t+1} \equiv R^f + (R_{t+1} - R^f)\alpha_{j,t} \tag{6}$$

Note that since $R^p_{j,t+1}$ is a weighted average over a constant R^f and a log-normal variable R_t, it is not itself log-normal. We follow Campbell (1993) and proceed with a log-linear approximation to the relationship between log portfolio returns and log individual-asset returns, taken about the point with zero excess returns:[10]

$$\log(R^p_{j,t+1}) \approx r^f + \alpha_{j,t}(r_{t+1} - r^f) + \frac{1}{2}\alpha_{j,t}(1 - \alpha_{j,t})\sigma^2 \tag{7}$$

where r_{t+1}, r^f denote the log returns on the risky and safe assets respectively. With this approximation, the budget constraint (4) becomes:

$$w_{j+1} = (w_j - c_{j,t}) \exp\left[r^f + \alpha_{j,t}(r_{t+1} - r^f) + \frac{1}{2}\alpha_{j,t}(1 - \alpha_{j,t})\sigma^2\right] \tag{8}$$

Solution Without Ambiguity: If there is no ambiguity, the expectations operator $E_{j,t}$ coincides with expectations formed under the objective probability distribution of returns. Proposition 1 gives the optimal consumption and portfolio allocations in this case.

Proposition 1 *Solving the household optimization (3) subject to (8) and (5), without ambiguity aversion, implies a value function of the form:*

$$V_j(w_j) = A_j \frac{w_j^{1-\gamma}}{1-\gamma} \tag{9}$$

where

$$A_j = \begin{cases} \left(\frac{1+b_{j+1}}{b_{j+1}}\right)^{\gamma} & \text{for } j = 0, ..., J-1 \\ 1 & \text{for } j = J \end{cases} \tag{10}$$

$$b_{j+1} = \left[\beta\phi_j A_{j+1}[1 + (1-\gamma)r^f + \frac{1}{2}(1-\gamma)\frac{(\mu - r^f)^2}{\gamma\sigma^2}]\right]^{-\frac{1}{\gamma}} \tag{11}$$

[10] This approximation is exact in the limit of continuous time (Campbell and Viceira, 2002).

Optimal consumption and portfolio choices are given by:

$$\alpha^* = \frac{\mu - r^f}{\gamma\sigma^2} \tag{12}$$

$$c^*_{j,t} = \begin{cases} \frac{b_{j+1}}{1+b_{j+1}} w_j & \textit{for } j < J \\ w_j & \textit{for } j = J \end{cases} \tag{13}$$

Proof. Appendix A ■

As is well-known, this type of problem implies that the proportion of the agent's portfolio invested in risky assets is constant over time and age (Campbell and Viceira, 2002). However, the path of optimal consumption depends on future survival probabilities, and therefore varies over the life-cycle.

2.3 Ambiguity Aversion

We consider the case in which there is ambiguity over the mean of the return on the risky asset, as in Peijnenburg (2018). Formally, while equation (6) accurately reflects the return an agent would receive on a given portfolio invested in period t, the agent considers a set of models under which returns are distorted away from this by an amount $\nu_{j,t}$:

$$R^p_{j,t+1} = R^f + (R_{t+1} - R^f + \sigma_1 \nu_{j,t})\alpha_{j,t} \tag{14}$$

where σ_1 is the standard deviation of R_{t+1}:

$$\sigma_1 = \exp(\mu)\left(\exp(\sigma^2) - 1\right)^{\frac{1}{2}} \tag{15}$$

To model the agent's aversion to ambiguity, we follow Hansen and Sargent (2008) and rewrite their dynamic programming problem as:

$$V^\theta_j(w_j) = \max_{c_{j,t},\alpha_{j,t}} \min_{\nu_{j,t}} \left\{ \frac{c^{1-\gamma}_{j,t}}{1-\gamma} + \beta\phi_j \left[\frac{1}{2\theta}\nu^2_{j,t} + E_{j,t}[V^\theta_{j+1}(w_{j+1})] \right] \right\} \tag{16}$$

subject to $w_J \geq 0$, the budget constraint (4), and the distorted returns process (14).

That is, the agent makes consumption and portfolio decisions based on a distorted law of motion for their assets, in which returns on the risky asset are systematically biased towards models for the risky return which deliver low continuation values in their optimization problem. In this way they make choices which are robust to their uncertainty over the process for risky asset returns.

However, the agent does not entertain an infinite set of models. Rather, they choose the distortion in the returns process behind their consumption and portfolio choices so that it minimizes expected utility, *plus* a cost of $\frac{1}{2\theta}\nu_{j,t}^2$. Intuitively, the parameter θ controls the agent's preference for robustness: larger values of θ imply the agent entertains larger deviations from the true returns process in equation (6). For a detailed discussion of this approach to modeling ambiguity aversion, see Hansen and Sargent (2008) and the survey in Ilut and Schneider (2022).

This formulation means that agents consider larger distortions if their value functions are more sensitive to model misspecification; in these cases the need for robustness is greater. Since value functions differ by age, the distortions to beliefs about risky asset returns due to ambiguity aversion will also vary across the life-cycle. Intuitively, although all agents share the same preference for robustness (they have the same θ), agents of different ages have different levels of exposure to changes in the return on risky assets.

Optimal Belief Distortion: We begin by solving the inner minimization problem, in which the agent chooses how much to distort their return expectations towards the 'worst case scenario'. In this, the following result is helpful.

Lemma 1 *Taking the same log-linear approximation approach as in* (7) *to the distorted returns in* (14)*, we can write*

$$E_{j,t}[V_{j+1}^{\theta}(w_{j+1})] \approx E_{j,t}[V_{j+1}^{\theta}(w_{j+1}^*)] + \frac{A_{j+1}}{1-\gamma}(w_j - c_{j,t})^{1-\gamma}(1-\gamma)\sigma\alpha_{j,t}\nu_{j,t} \tag{17}$$

where

$$w_{j+1}^* = (w_j - c_{j,t})[R^f + \alpha_{j,t}(R_{t+1} - R^f)] \tag{18}$$

is the next-period wealth the agent would achieve under the central model without return distortions.

Proof. Appendix A ■

Substituting this into the Bellman equation (16), it is then straightforward to obtain the first order condition for the inner minimization problem:

$$\nu_{j,t} = -\theta A_{j+1}(w_j - c_{j,t})^{1-\gamma}\sigma\alpha_{j,t} \tag{19}$$

Consumption and Portfolio Allocation: Substituting the optimal distortion (19) into equation (16), the household chooses consumption and the share of savings invested in risky assets to solve:

$$V_j^\theta(w_j) = \max_{c_{j,t},\alpha_{j,t}} \left\{ \frac{c_{j,t}^{1-\gamma}}{1-\gamma} + \beta\phi_j \left[-\frac{1}{2}\theta A_{j+1}^2 (w_j - c_{j,t})^{2-2\gamma}\sigma^2\alpha_{j,t}^2 + E_{j,t}[V_{j+1}^\theta(w_{j+1}^0)] \right] \right\} \tag{20}$$

Proposition 2 characterizes the solution.

Proposition 2 *The value function takes the form:*

$$V_j^\theta(w_j) = A_j \frac{w_j^{1-\gamma}}{1-\gamma} + \theta B_j \frac{w_j^{2(1-\gamma)}}{2(1-\gamma)} + O(\theta^2) \tag{21}$$

where B_j is an age-dependent combination of model parameters, defined in Appendix A.

In the approximate solution where we drop terms in θ^2, optimal portfolio choice and consumption are given by:

$$\alpha_{j,t} = \alpha^* + \theta\alpha^* w_j^{1-\gamma}\Omega_{\alpha j} \tag{22}$$

$$c_{j,t} = c_{j,t}^* + \theta w_j^{2-\gamma}\Omega_{cj} \tag{23}$$

where $\alpha^*, c_{j,t}^*$ are the solutions without ambiguity defined in Proposition 1, and $\Omega_{\alpha j}, \Omega_{cj}$ are functions of $b_{j+1}, A_{j+1}, B_{j+1}$, defined in Appendix A.

Proof. Appendix A ■

With $\theta = 0$, we therefore return to the standard expected-utility solution (Proposition 1). With some ambiguity aversion ($\theta > 0$), however, both portfolio and consumption decisions shift away from this solution. Importantly, the effect of ambiguity aversion depends on both the agent's wealth and, through $\Omega_{\alpha j}, \Omega_{cj}$, their expected future lifespan. Intuitively, this occurs because agents of different ages are differentially exposed to the return on risky assets, and so opt for different levels of belief distortion in response to their ambiguity aversion. As the resulting distortions to beliefs, consumption, and portfolio shares are generally nonlinear functions of wealth and demographics, we now turn to a simple special case to explore the mechanisms analytically, before returning to this full model in Section 4.

3 Results with Three-Period Lifespans

We now explore the mechanisms relating aging, life expectancy, and ambiguity to portfolio decisions by restricting the model to $J = 2$. Agents therefore live for a maximum of three periods: young ($j = 0$), middle aged ($j = 1$), and old ($j = 2$). Furthermore, we assume that $\phi_0 = 1$, so all agents survive at least to middle age. In this simple context, population aging therefore only occurs through an increase in ϕ_1, the probability of surviving to old age.

3.1 Consumption and Portfolio Allocation: No Ambiguity Case

Consider an agent born in period t. To understand the effects of ambiguity aversion in this environment, it is helpful to first examine the forces that drive consumption and saving in the baseline model without ambiguity. In this case, the portfolio share in risky assets is constant, as in equation (12). Applying the remaining elements of Proposition 1, we obtain closed-form solutions for consumption in each period of the agent's life.

Proposition 3 *An agent with $\phi_0 = 1, \phi_2 = 0$ and initial wealth w_0 chooses consumption when young and middle-aged according to:*

$$c_{0,t} = \frac{\tilde{b}^2}{\phi_1^{\frac{1}{\gamma}} + \tilde{b}(1+\tilde{b})} w_0 \tag{24}$$

$$c_{1,t+1} = \frac{\tilde{b}}{\phi_1^{\frac{1}{\gamma}} + \tilde{b}(1+\tilde{b})} R^p_{0,t+1} w_0 \tag{25}$$

where $\tilde{b}$ is a strictly positive combination of age-independent parameters:

$$\tilde{b} = \left[\beta\left(1 + (1-\gamma)r^f + \frac{1}{2}\frac{(1-\gamma)}{\gamma}(\alpha^*)^2\right)\right]^{-\frac{1}{\gamma}} \tag{26}$$

Conditional on surviving to old age, they then have:

$$c_{2,t+2} = \frac{\phi_1^{\frac{1}{\gamma}}}{\phi_1^{\frac{1}{\gamma}} + \tilde{b}(1+\tilde{b})} R^p_{0,t+1} R^p_{1,t+2} w_0 \tag{27}$$

Proof. Appendix A ■

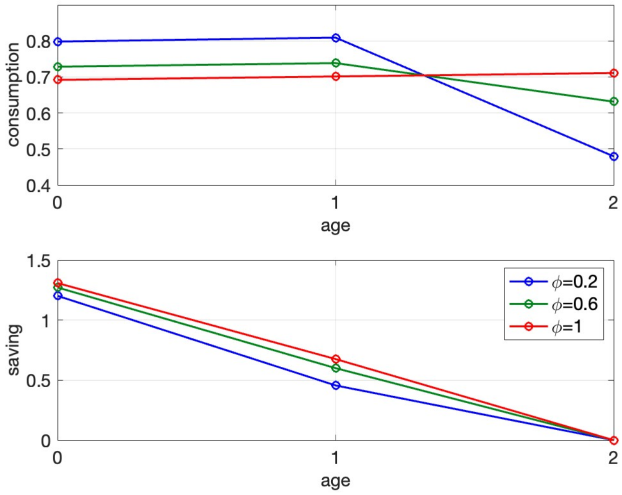
As the probability of surviving to old age (ϕ_1) increases, the incentive to save for consumption in old age rises, so agents consume less both when they are young and when they are middle aged ($c_{0,t}$ and $c_{1,t+1}$ decrease). For agents that do survive to old age, a greater ϕ_1 implies higher consumption $c_{2,t+2}$, due to the extra savings built up earlier in life. Together, these results imply that savings are depleted less quickly through the life cycle if average life expectancies are longer.[11]

These patterns are displayed in Figure 1, which plots the paths of consumption and saving over the life-cycle for three different values of ϕ_1. When the probability of sur-

[11]This is consistent with Foltyn and Olsson (2021), who find that individuals with longer subjective life expectancies accumulate more wealth over their life-cycle than those who expect to die earlier.

viving to old age is low, agents consume a lot in their youth and middle age. If they do survive to old age, they therefore experience a large consumption drop. With a greater survival probability, young and middle-aged consumption is lower, and the subsequent consumption fall in old age is lower.[12]

Figure 1: Consumption and saving paths with no ambiguity.



Note: Plots constructed using $J = 2$, $\mu = 0.06$, $r^f = 0.045$, $\sigma = 0.1$, $\gamma = 3$, $\beta = 0.99$, $\phi_1 = 0.2, 0.6, 1$, $w_0 = 2$, and risky asset returns set to their expected level every period. This therefore abstracts from return shocks.

3.2 Ambiguity Aversion

We now add ambiguity aversion back into the model. The key element of this model is how the distortion in return expectations due to ambiguity aversion varies with age, wealth, and the probability of surviving to old age. These distortions are given in Proposition 4.

Proposition 4 *The optimal distortion in beliefs about risky asset returns for an agent with wealth w_t and age j is:*

$$\nu_{0,t} = -\frac{\theta\sigma\alpha^*(\phi_1^{\frac{1}{\gamma}} + \tilde{b})}{\tilde{b}^\gamma(\phi_1^{\frac{1}{\gamma}} + \tilde{b} + \tilde{b}^2)^{1-\gamma}} w_0^{1-\gamma} \tag{28}$$

$$\nu_{1,t} = -\frac{\theta\sigma\alpha^*\phi_1^{\frac{1-\gamma}{\gamma}}}{(\phi_1^{\frac{1}{\gamma}} + \tilde{b})^{1-\gamma}} w_1^{1-\gamma} \tag{29}$$

$$\nu_{2,t} = 0 \tag{30}$$

[12]With $\phi_1 = 1$ the consumption path is slightly increasing over time here due to precautionary saving. $c_{2,t+2} > c_{1,t+1}$ in the model whenever $\phi_1 > (\tilde{b}/R^p_{1,t+2})^\gamma$, which is close to 1 for most calibrations.

Proof. Appendix A ■

To understand the implications of these distortions, we first compare agents with the same wealth but different ages, to isolate the effects of age and survival probabilities. We then go on to analyse the interactions with varying wealth.

3.2.1 Age Effects

First, Proposition 4 implies that old agents ($j = 2$) do not distort their beliefs at all (30). This is because they save nothing, and so have no exposure to asset returns. There is no need for them to make their decisions robust to doubts about average asset returns. Similarly, note that if $\phi_1 = 0$ then a middle-aged agent sets $\nu_{1,t} = 0$, for the same reason: they will die for certain at the end of the period, so they do not save and are not exposed to ambiguity. In all other cases $\nu_{j,t} < 0$, so the agents distort their beliefs towards lower returns on the risky asset.

Corollary 1 shows a further equivalence between two other extreme special cases:

Corollary 1 *Let $\nu_{j,t}(\phi)$ be the distortion chosen by an agent of age j facing a survival probability of $\phi_1 = \phi$. Then, if $w_0 = w_1$:*

$$\nu_{0,t}(0) = \nu_{1,t}(1) \tag{31}$$

Proof. Appendix A ■

That is, a young agent who will die for certain after middle age distorts beliefs in the same way as a middle-aged agent who will survive to old age for certain. In both cases, the agent knows they have one more period of consumption, and so behavior is the same for each. This highlights that life *expectancy*, rather than age, is the critical factor in how ambiguity aversion affects beliefs in this environment.

Second, Proposition 4 also implies that changes in ϕ_1 affect the belief distortions among young and middle-aged agents away from these edge cases.

Corollary 2 *As ϕ_1 changes, then holding wealth constant the optimal belief distortions of young agents are such that:*

$$\frac{\partial \nu_{0,t}}{\partial \phi_1} < 0 \tag{32}$$

Holding wealth constant, the distortions of middle-aged agents are such that:

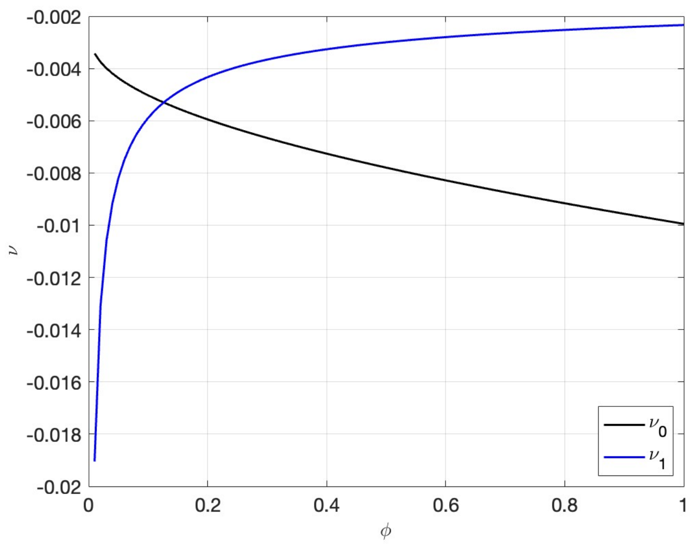
$$\frac{\partial \nu_{1,t}}{\partial \phi_1} \begin{cases} < 0 & \text{if } \gamma < 1 \\ = 0 & \text{if } \gamma = 1 \\ > 0 & \text{if } \gamma > 1 \end{cases} \tag{33}$$

Proof. Appendix A ■

As the probability of surviving to old age rises, young households distort their beliefs more strongly, becoming more pessimistic about equity returns. If the EIS is greater than 1 ($\gamma < 1$), middle-aged households do the same. However, if the EIS is less than 1 ($\gamma > 1$), they decrease the magnitude of their distortion.

This divergence is at the heart of our mechanism: in the empirically plausible case with $\gamma > 1$, as life expectancy rises the young get more pessimistic about equity returns, while older agents get more optimistic. The effect is shown in Figure 2.

Figure 2: Belief distortions as ϕ_1 varies.



Note: Plots constructed using $J = 2$*,* $\theta = 0.045$*,* $\mu = 0.06$*,* $r^f = 0.045$*,* $\sigma = 0.1$*,* $\gamma = 3$*,* $\beta = 0.99$*,* $\phi_1 \in (0, 1]$*,* $w_0 = w_1 = 2$*, and risky asset returns set to their expected level every period. This therefore abstracts from the effect of return shocks.*

At ϕ_1 close to 0, young households are less pessimistic than middle-aged households. As the survival probability grows, these positions reverse.

To understand the mechanisms driving the divergent responses to increasing longevity, we return to the first order condition for the inner minimization in equation (16). The agent chooses the degree to which they distort their return expectations by balancing the marginal damage to expected continuation values with the marginal penalty to considering a larger distortion:

$$\frac{\partial}{\partial \nu_{j,t}} \left\{ \frac{1}{2\theta} \nu_{j,t}^2 + E_{j,t}[V_{j+1}^{\theta}(w_{j+1})] \right\} = 0 \tag{34}$$

Equation (19) is then simply the result of combining this with Lemma 1 and rearranging. However, we can alternatively write this condition as:

$$\nu_{j,t} = -\theta \frac{\partial E_{j,t}[V_{j+1}^{\theta}(w_{j+1})]}{\partial \nu_{j,t}} \tag{35}$$

$$\approx -\theta\sigma\alpha^* \underbrace{(w_j - c_{j,t})}_{\text{Wealth Channel}} \cdot \underbrace{\frac{\partial E_{j,t}[V_{j+1}^{\theta}(w_{j+1})]}{\partial w_{j+1}}}_{\text{Marginal Utility Channel}} \tag{36}$$

where the second line uses the same approximation as in Lemma 1.

The distortion is set proportional to the sensitivity of expected continuation values to asset returns. Intuitively, the more exposed the agent is to changes in risky asset returns, the more they wish to make their decisions robust to ambiguity over those returns. That sensitivity can be broken down into two channels: the wealth channel, and the marginal utility channel.

The wealth channel operates because when an agent saves more for the next period, their next-period wealth is more strongly affected by asset returns. In other words, they have more skin in the game. As discussed in Section 3.1, when ϕ_1 increases both young and middle-age agents increase their saving.[13] For both young and middle-age agents, this channel therefore implies greater belief distortions when ϕ_1 rises.

The marginal utility channel operates because a given decrease in asset returns will have a larger effect on utility for an agent with a large marginal utility of wealth in the following period. Through a standard envelope theorem, the marginal utility of wealth in period $t+1$ is equal to the marginal utility of consumption in $t+1$. Since our model features a diminishing marginal utility of consumption, this channel will be more powerful when next-period consumption is expected to be low.

This channel is what drives the divergence in beliefs across cohorts. Recall from Section 3.1 that, as ϕ_1 increases, the consumption of middle-aged agents falls, while the consumption of old agents rises. Agents who are currently young therefore expect to have a greater marginal utility of wealth in the following period, when they will be middle-aged. An increase in ϕ_1 makes them more sensitive to changes in wealth, increasing the strength of the marginal utility channel. In contrast, current middle-aged agents expect

[13] The paths of consumption and saving remain qualitatively unchanged with the introduction of ambiguity aversion, as we typically consider small values of θ. Sufficient conditions for this result, and a numerical example, are provided in Appendix B.

to have more wealth in future, and so a lower marginal utility, implying a smaller marginal utility channel.

For a young agent, both the wealth and marginal utility channels therefore imply that they become more pessimistic when ϕ_1 rises. For a middle-aged agent, the channels act in opposite directions. To see which dominates, note that for a middle-aged agent we obtain:[14]

$$\frac{\partial E_{1,t}[V_2^{\theta}(w_2)]}{\partial w_2} \propto (w_1 - c_{1,t})^{-\gamma} \tag{37}$$

Substituting this into equation 36 implies:

$$\nu_{1,t} \propto -\theta\sigma\alpha^*(w_1 - c_{1,t})^{1-\gamma} \tag{38}$$

For a middle-aged agent, an increase in ϕ_1 implies a rise in $w_j - c_{j,t}$. With $\gamma < 1$, the wealth channel dominates, and middle-aged agents therefore increase the magnitude of their belief distortions when survival probabilities rise ($\nu_{j,t}$ becomes more negative). In the empirically plausible case with $\gamma > 1$, however, the marginal utility channel dominates, and middle-age agents become more optimistic about returns. With log utility ($\gamma = 1$) the effects cancel out and middle-aged agents do not adjust $\nu_{1,t}$ with ϕ_1.

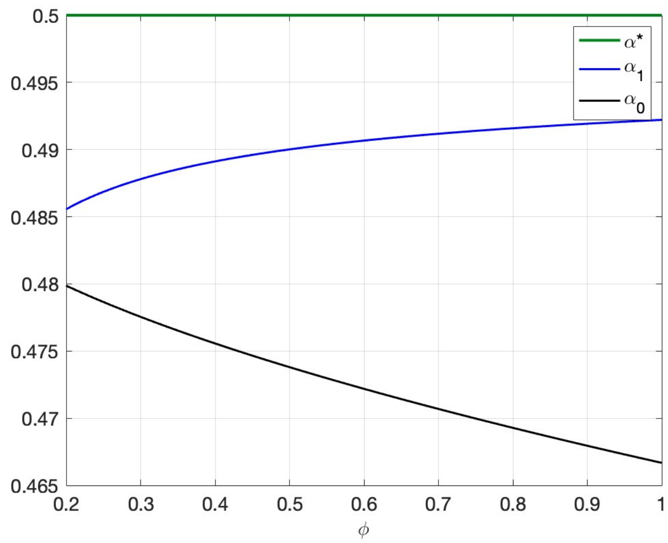
Portfolio Allocation: Figure 3 plots the share of agent portfolios invested in the risky asset as ϕ_1 changes, for the same parameters as Figure 2. Both young and middle-aged agents allocate lower shares of their wealth to risky assets than they would in the absence of ambiguity, as in other models in the literature (Garlappi et al., 2007; Campanale, 2011) and consistent with empirical evidence (Dimmock et al., 2016). This is a direct consequence of Proposition 4: the belief distortions due to ambiguity aversion imply the agent acts as if the risky asset has a lower expected return than its true mean, and so invests less in that asset than they would in the absence of ambiguity. For $\phi_1 > 0.2$ in this calibration, young agents distort their return beliefs more in response to ambiguity than middle-aged agents, so their risky asset share is correspondingly lower. We find the same qualitative pattern in the quantitative illustration in Section 4.

The changes in risky asset shares as the population ages (ϕ_1 increases) follow from Corollary 2. As ϕ_1 increases along the x-axis, middle-aged agents reduce the distortions in their beliefs, increasing their risky asset share towards the benchmark share without ambiguity (α^*). In contrast, young households increase their distortions, and move further from this benchmark level.

Note that the models in Campanale (2011) and Peijnenburg (2018) also feature risky

[14]This follows from Proposition 2, and the fact that $A_2 = 1$.

Figure 3: Risky asset shares without ambiguity (α_{BM}), and with ambiguity for young (α_0) and middle-aged (α_1) agents, as ϕ_1 varies.



Note: Plots constructed using $J = 2$, $\theta = 0.045$, $\mu = 0.06$, $r^f = 0.045$, $\sigma = 0.1$, $\gamma = 3$, $\beta = 0.99$, $\phi_1 \in (0, 1]$, $w_0 = w_1 = 2$, *and risky asset returns set to their expected level every period. This therefore abstracts from the effect of return shocks.*

asset shares that increase with age ($\alpha_{1,t} > \alpha_{0,t}$). However, the mechanisms in those papers are different from ours: in both, agents learn over time from observed realisations of asset returns, gradually reducing the set of models they are willing to consider. In our framework, that would entail a fall in θ as agents progress from young to middle-aged, independently of survival probabilities. In contrast, we keep θ constant, but allow the optimal distortion to vary with agent exposure to asset returns. The effect of survival probabilities on the age-profile of asset shares, through the wealth and marginal utility channels, is therefore unique to our mechanism.

3.2.2 Wealth Effects

The first order condition for belief distortions in equation (36) highlights that, just as with age, the effects of an increase in wealth depend on the wealth and marginal utility channels.

With an increase in wealth, these channels act in opposite directions. The wealth channel implies larger belief distortions for wealthier households, as they save more, so have more exposure to asset returns. The marginal utility channel implies the opposite: wealthier households have smaller belief distortions, because their continuation values are less sensitive to marginal changes in future wealth. As with the effect of survival probabilities on middle-aged agents, which channel dominates depends on whether the

EIS is greater or less than 1.

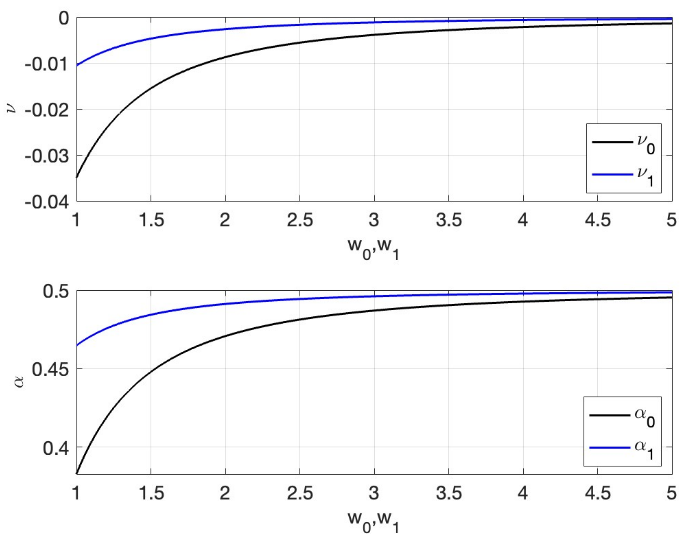
Corollary 3 *As w_j changes, then optimal belief distortions of young and middle-aged agents are such that for $j = 0, 1$:*

$$\frac{\partial \nu_{j,t}}{\partial w_j} \begin{cases} < 0 & \text{if } \gamma < 1 \\ = 0 & \text{if } \gamma = 1 \\ > 0 & \text{if } \gamma > 1 \end{cases} \tag{39}$$

Proof. Appendix A ■

Under our preferred calibrations ($\gamma > 1$), being wealthier causes agents to become more optimistic about asset returns. As a result, wealthier agents invest a greater share of their wealth in the risky asset. These patterns are shown in Figure 4. Although our model does not feature non-participation, the direction of this effect is consistent with evidence in Briggs et al. (2020) that exogenous increases in wealth increase the probability that households invest in equities.

Figure 4: Belief distortions and risky asset shares vary with wealth.



Note: Plots constructed using $J = 2$, $\theta = 0.045$, $\mu = 0.06$, $r^f = 0.045$, $\sigma = 0.1$, $\gamma = 3$, $\beta = 0.99$, $\phi_1 = 0.7$, w_0 and w_1 vary from 1 to 5, and risky asset returns set to their expected level every period. This therefore abstracts from the effect of return shocks.

Interactions with Age Effects: As well as directly affecting belief distortions as in Corollary 3, an agent's wealth can affect the strength of the age effects on beliefs studied in Section 3.2.1. In our preferred parameter region of $\gamma > 1$, when wealth is higher, age effects are smaller in magnitude, for agents of all ages.

Corollary 4 *As w_j changes, the age effects on optimal belief distortions are such that for $j = 0, 1$:*

$$sign\left\{\frac{\partial}{\partial w_j}\left(\frac{\partial \nu_j}{\partial \phi_1}\right)\right\} = \begin{cases} sign\left\{\dfrac{\partial \nu_j}{\partial \phi_1}\right\} & if\, \gamma < 1 \\ 0 & if\, \gamma = 1 \\ -sign\left\{\dfrac{\partial \nu_j}{\partial \phi_1}\right\} & if\, \gamma > 1 \end{cases} \tag{40}$$

This implies that if $\gamma > 1$, the effect of ϕ_1 on optimal belief distortions decreases in magnitude as w_j increases.

Proof. Appendix A ■

Intuitively, at high levels of wealth, future marginal utility is less sensitive to changes in returns, and so the marginal utility channel is weakened. When $\gamma > 1$, the marginal utility channel is the dominant channel determining how belief distortions $\nu_{j,t}$ change with ϕ_1 at all ages. Weakening that channel therefore weakens the effects of ϕ_1 on beliefs.

3.3 Intergenerational Inequality

We now use the results developed above to analyse the impacts of increased longevity through ambiguity aversion. The first implication we study is on intergenerational wealth inequality.

Using equation 8, we can express the ratio between wealth at ages $j + 1$ and j as:

$$\frac{w_{j+1}}{w_j} = \left(1 - \frac{c_j}{w_j}\right) \exp\left(r^f + \alpha_{j,t}(r_{t+1} - r^f) + \frac{1}{2}\sigma^2 \alpha_{j,t}(1 - \alpha_{j,t})\right) \tag{41}$$

Expanding out c_j and α_j using Proposition 2, this becomes:

$$\begin{aligned}\frac{w_{j+1}}{w_j} = \left(\frac{1}{1 + b_{j+1}} - \theta w_j^{1-\gamma}\Omega_{c,j}\right) \exp\Bigg(r^f + \alpha^*(r_{t+1} - r^f) + \frac{1}{2}\sigma^2\alpha^*(1 - \alpha^*) \\ + \theta\alpha^* w_j^{1-\gamma}\Omega_{\alpha,j}\left[r_{t+1} - r^f + \frac{1}{2}\sigma^2(1 - 2\alpha^* - \theta\alpha^* w_j^{1-\gamma}\Omega_{\alpha,j})\right]\Bigg)\end{aligned} \tag{42}$$

Finally, using the definition of b_{j+1} for $j = \{0, 1\}$ (Appendix A) note that:

$$\frac{1}{1 + b_1} = \frac{\phi_1^{\frac{1}{\gamma}} + \tilde{b}}{\phi_1^{\frac{1}{\gamma}} + 2\tilde{b}} \tag{43}$$

$$\frac{1}{1 + b_2} = \frac{\phi_1^{\frac{1}{\gamma}}}{\phi_1^{\frac{1}{\gamma}} + \tilde{b}} \tag{44}$$

both of which are strictly increasing in ϕ_1.

In the absence of ambiguity ($\theta = 0$), only the first term in equation 42 changes with ϕ_1. As the survival probability rises, agents save more for the future, so middle-aged agents become wealthier relative to young agents, and similarly old agents become wealthier relative to middle-aged agents.

Ambiguity adds two further channels to this change in wealth across the life-cycle. First, a rise in ϕ_1 affects agent portfolio choices, and so affects average returns. This generates the terms in square brackets in equation 42. In Section 3.2.1 we showed that rising ϕ_1 has opposing effects on the risky asset shares of young and middle-aged agents in the empirically plausible case of $\gamma > 1$. Young agents reduce α_0, which reduces the wealth of the middle-aged relative to the young. Middle-aged agents increase α_1, increasing the relative wealth of the old. Through this channel, an aging population leads to a greater wealth concentration among older households.

Second, ambiguity also affects the amount saved by each agent, through the term $-\theta w_j^{1-\gamma}\Omega_{c,j}$. For both young and middle-aged agents, $\Omega_{c,j}$ varies with ϕ_1, and for middle-aged agents so does w_j. In Appendix B we show that this term is potentially non-monotonic in ϕ_1, so it has an ambiguous effect on wealth inequality. However, in the simple calibration used throughout this section, and in the quantitative model in Section 4, this effect is negligible relative to the other channels.[15]

3.4 Aggregation

Next, we study how aging affects the composition of aggregate asset demand.

All agents within a cohort are identical. The aggregate demand for safe and risky assets is therefore given by:

$$AD(safe) = (1 - \alpha_0)(w_0 - c_0) + (1 - \alpha_1)(w_1 - c_1) \tag{45}$$

$$AD(risky) = \alpha_0(w_0 - c_0) + \alpha_1(w_1 - c_1) \tag{46}$$

where we have used the result that old agents do not save in either asset. Note that this implies the composition effects of aging, as studied in e.g. Auclert et al. (2021), are absent here: the age-composition of *asset market participants* is constant as ϕ_1 rises. This simplification allows us to focus on the novel channels introduced by ambiguity aversion here. We relax this in Section 4.

[15]Specifically, with the calibration used in Figure 2, an increase in ϕ_1 from 0.2 to 0.8 implies that w_2/w_1 and w_1/w_0 increase by 31.02% and 7.48% respectively. The change in consumption due to ambiguity accounts for 0.014% and 0.019% of those changes.

Recall that if there is no ambiguity aversion ($\theta = 0$), risky asset shares α_j are constant, and both young and middle-aged agents cut consumption when ϕ_1 rises (Proposition 3). As a result, the aggregate demand for both types of asset rises, as longer life expectancy encourages greater saving for old age.

In the case with ambiguity aversion, the deviation of α_j and c_j from the no-ambiguity benchmark is proportional to the degree of ambiguity aversion θ (Proposition 2). For small θ, we therefore maintain the result that $AD(safe)$ and $AD(risky)$ rise with ϕ_1, as in the no-ambiguity benchmark.

Ambiguity aversion does, however, affect the speed at which each aggregate asset demand rises, which therefore affects the *composition* of asset demand as the population ages. This is displayed in equation 47, which gives the population analogue of the individual-level risky share α_j.

$$\frac{AD(risky)}{AD(safe) + AD(risky)} = \frac{\alpha_0(w_0 - c_0) + \alpha_1(w_1 - c_1)}{w_0 - c_0 + w_1 - c_1} \tag{47}$$

Substituting out for the individual risky asset shares α_j using equation 22, this becomes:

$$\frac{AD(risky)}{AD(safe) + AD(risky)} = \alpha^* + \theta\alpha^* \left(\frac{\Omega_{\alpha,0} w_0^{1-\gamma}(w_0 - c_0) + \Omega_{\alpha,1} w_1^{1-\gamma}(w_1 - c_1)}{w_0 - c_0 + w_1 - c_1} \right) \tag{48}$$

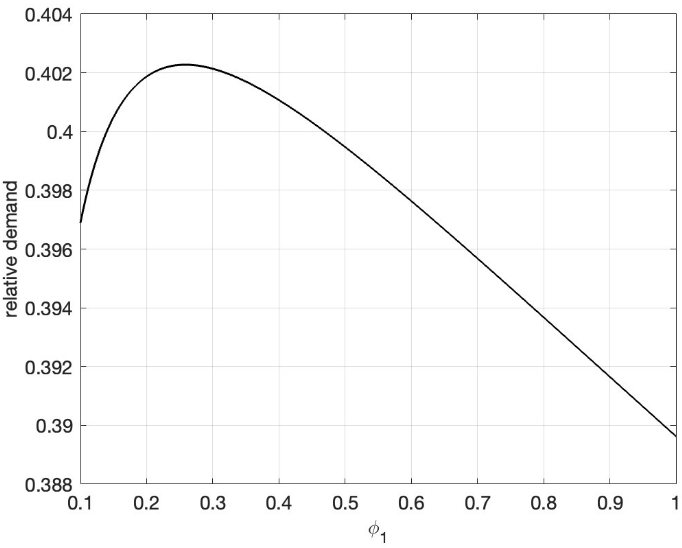
In the absence of ambiguity, the relative demand for risky assets is a constant (α^*). However, with ambiguity ($\theta > 0$), demand for safe and risky assets are no longer in fixed proportions, and the relative demand for each asset changes with the survival probability.

These relative demand changes are shown in Figure 5. The no-ambiguity relative demand is constant at α^*, which with this calibration is equal to 0.5. As in e.g. Garlappi et al. (2007), ambiguity aversion therefore reduces the relative demand for risky assets below this level. The contribution of our model is that we can ask how that relative demand changes with survival rates. As ϕ_1 rises, the demand for risky assets relative to safe assets follows a hump-shape: it rises, reaches a peak, then falls.

This hump-shape in relative risky asset demand occurs because young and middle-aged agents shift their belief distortions in different directions as ϕ_1 increases, as shown in Section 3.2.1. Corollary 2 shows that as the survival probability rises, the young become more pessimistic about asset returns, while the middle-aged become more optimistic. As a result, young agents decrease their relative demand for risky assets, while middle-aged agents increase their relative demand.

When the probability of surviving to old age is low, young households save only a

Figure 5: Relative asset demand $\frac{AD(risky)}{AD(safe)+AD(risky)}$ varies with ϕ_1.



Note: Plots constructed using $J = 2$, $\mu = 0.06$, $r^f = 0.045$, $\sigma = 0.1$, $\theta = 0.045$, $\gamma = 3$, $\beta = 0.99$, $\phi_1 \in (0, 1]$, $w_0 = 1$, *and risky asset returns set to their expected level every period. This therefore abstracts from the effect of return shocks.*

small fraction of their endowment (Proposition 3, extended to the ambiguity case in Appendix B). As a result, when they become middle-aged, they have little wealth: w_1 is low relative to w_0. Corollary 4 then implies that any small increase in ϕ_1 has a stronger effect on the beliefs of middle-aged than young agents. Initially, the middle-aged agents react most strongly to ϕ_1, and relative risky asset demand rises.

However, as ϕ_1 rises, young agents increase their saving, and these mechanisms work in reverse. Wealth w_1 rises, and so age effects become weaker for middle-aged agents, ultimately becoming smaller than the effects on young agents. At high ϕ_1, further aging of the population therefore implies a fall in relative risky asset demand.

Finally, note that relative risky asset demand is also affected by a composition channel. As ϕ_1 rises, w_1 rises, which means that middle-aged agents account for a greater share of aggregate saving. Since for most values of ϕ_1 middle-aged agents are more optimistic than young agents (Figure 2), this also causes the aggregate relative risky asset demand to rise with ϕ_1, shifting the peak in Figure 5 to the right.

3.5 Endogenous Equity Premium

So far, we have considered a small open economy aging alone. In that case, the variation in relative demand for safe and risky assets shown in Figure 5 does not affect returns or asset prices. However, in a closed economy, or indeed in a world where all countries are aging simultaneously, this will no longer be true.

We therefore extend the model here, and instead assume that the relative supply of safe and risky assets is fixed. This allows us to study the effects of aging on the equity premium $\mu - r^f$, as this must adjust to ensure that asset markets clear. The equity premium is particularly of interest because it controls how much heterogeneity there is between the wealth accumulation of agents with different beliefs. It is therefore central to how our mechanisms will affect intergenerational inequality.

Specifically, let $S_t(safe)$ and $S_t(risky)$ denote the supply of safe and risky assets in period t. The relative supply of risky assets is assumed to be fixed at $\bar{S}$:

$$\frac{S_t(risky)}{S_t(safe) + S_t(risky)} = \bar{S} \tag{49}$$

For asset markets to clear, we therefore require:[16]

$$\frac{AD_t(risky)}{AD_t(safe) + AD_t(risky)} = \bar{S} \tag{50}$$

This particular assumption on asset supply is useful here, because it implies that if there is no ambiguity aversion, the solution is trivial. From equation 48, if $\theta = 0$ then the relative demand for risky assets is constant at α^*, which itself is directly proportional to the equity premium (equation 12). In this case, the equilibrium equity premium is therefore a constant, unaffected by changes in survival probabilities:

$$(\mu - r^f | \theta = 0) = \gamma \sigma^2 \bar{S} \tag{51}$$

As a result, any dependence of the equity premium on ϕ_1 must come through ambiguity aversion. In this way, our equity premium analysis is similar in spirit to our analysis of the small open economy above, in which case individual portfolio choices are independent of ϕ_1 unless there is some ambiguity over risky returns.

To analyse the equity premium in the case with ambiguity, it is useful to first note that relative aggregate demand increases monotonically in the equity premium.

Lemma 2 *For any $\theta < \theta^*$:*

$$\frac{\partial}{\partial \mu} \left(\frac{AD_t(risky)}{AD_t(safe) + AD_t(risky)} \right) > 0 \tag{52}$$

$$\frac{\partial}{\partial r^f} \left(\frac{AD_t(risky)}{AD_t(safe) + AD_t(risky)} \right) < 0 \tag{53}$$

[16] This condition would still be necessary, though not sufficient, for asset market clearing in a model with fixed supplies of both assets individually.

where $\theta^* > 0$ is a threshold defined in Appendix A.

Proof. Appendix A ■

This is intuitive: as in the case without ambiguity, if the equity premium rises, then the expected return on risky assets rises relative to safe assets, rendering them more attractive to investors. Whether the equity premium rises because μ rises or r^f falls, the relative demand for risky assets therefore rises.

From this we derive two implications. First, Proposition 4 implies that for all values of ϕ_1, young and middle-aged agents distort their beliefs towards (weakly) lower risky asset returns. This is why ambiguity reduces the relative demand for risky assets, as shown in Figure 5. To offset this and ensure market clearing, the equity premium must therefore be higher than if there was no ambiguity, as in other models in this literature (e.g. Dimmock et al., 2016).

Second, the analysis in the previous sections highlights that individual and aggregate portfolio choices change with ϕ_1, implying that the equity premium will change as survival probabilities rise. Specifically, the equity premium is U-shaped in ϕ_1.

The intuition for this result follows directly from the discussion in Section 3.4. As ϕ_1 rises from a low level, then the relative aggregate demand for risky assets rises, as middle-aged agents become more optimistic about risky returns. This pushes the equity premium down, to clear asset markets. As ϕ_1 continues to rise, the relative aggregate demand for risky assets begins to fall, as increasing pessimism from young agents dominates the optimism from the middle-aged. That in turn implies the equity premium rises.

Interestingly, although the model is extremely simple, this is consistent with qualitative patterns in equity premia in the last 75 years. Since 1950, developed economies have experienced substantial rises in life expectancy. Over the same period, Kuvshinov and Zimmermann (2020) document that equity premia in developed economies have followed a U-shape, first falling, and then rising again after 1990.

4 Quantitative Illustration

We now return to the model in Section 2. We calibrate the model to survival probabilities, and the wealth distribution, in the US in 2019. The portfolio choices generated from the model match the age-profile of risky asset shares in the data. We then use the model to examine the effect of demographic change on portfolios in the coming decades. To do this, we compare the 2019 calibration with a counterfactual using projected survival probabilities for 2100. Even holding the distribution of wealth fixed, changes in life expectancy imply non-trivial changes in the age profile of portfolio composition.

4.1 Calibration

We calibrate the model such that one period is one year. First, we set some parameters to standard values in macroeconomics and finance, as displayed in Table 1.[17]

Table 1: Calibration block 1.

Parameter	Description	Value	Source
β	Discount factor	0.96	Standard
γ	Inverse EIS	1.8	Crump et al. (2022)
r^f	Risk-free rate	0.023	Cocco et al. (2005)
μ	Expected equity return	0.057	Kuvshinov and Zimmermann (2020)
σ	Std. deviation equity returns	0.18	Viceira (2001)

Second, we calibrate the survival probabilities using age-specific mortality rates in the US in 2019 reported by the US Office of the Chief Actuary at the Social Security Administration. This mortality data is representative for the US population. We specify the survival probability ϕ_j as one minus the mortality rate for people of age j. We also set the initial age to $j = 30$, and the maximum possible lifespan to $J = 109$: this is the first age in the data at which the annual mortality rate exceeded 50%.

The third step is to calibrate wealth holdings. We use the 2019 wave of the Survey of Consumer Finances (SCF) to obtain median financial assets by age in 2019. We use financial assets, rather than net worth, to be consistent with the commonly-reported risky asset shares from the SCF, which typically involve risky assets as a share of financial assets (see e.g. Chang et al., 2018).[18] We then set the wealth of each agent in period t to the median financial assets for their age group from the data. Formally, agents still behave as set out in Section 2. We simply assign each agent an unanticipated lump sum transfer at the start of each period such that wealth in the model matches that in the SCF.

This approach means that we are taking the exact age-profile of wealth in 2019 as an input to the model, rather than allowing it to emerge endogenously in equilibrium. The aim is to provide a quantification of the strength of the channels we study at that single point in time. As with the calibration of ϕ_j, we assign all households in a given age range the same wealth. Full details of this data and calibration are in Appendix C.1.

Finally, the only parameter left to calibrate is the preference for robustness parameter θ. Several papers have used ambiguity aversion to explain why risky asset shares are typically lower than in standard models (see e.g. Guidolin and Rinaldi, 2013, for a review

[17] The expected equity return is computed using the calibrated r^f and the average US equity premium over 1990-2015, as documented in Kuvshinov and Zimmermann (2020).

[18] There are only a small number of SCF participants above the age of 80, so we do not have reliable wealth data for households at those ages. Since we do not conduct any aggregation exercises in this section, this does not affect the results below. We simply cut off all age-profile plots at age 80, so this lack of data for the oldest households has no effect on any of the figures.

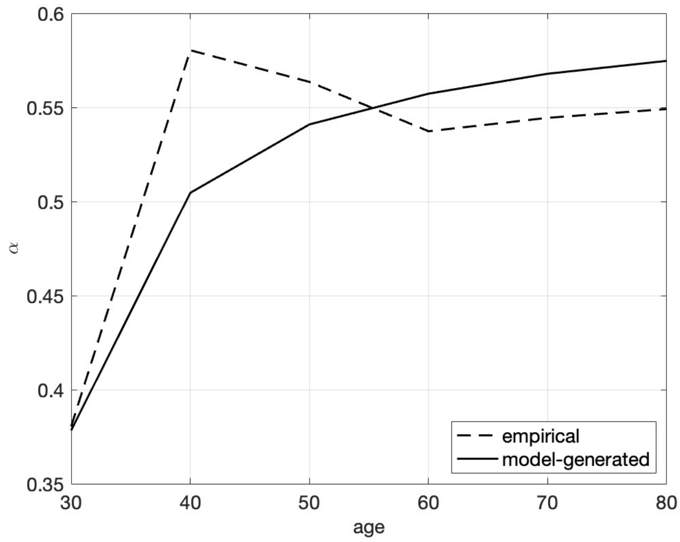
of the early literature). We therefore choose θ to target the average portfolio share in risky assets across all ages in the 2019 SCF (details in Appendix C.1).[19] We do not target any aspect of how that risky asset share varies across ages.

Targeting the average risky share implies a $\theta = 0.008$. In Appendix C.2 we show that this implies reasonable values of detection error probabilities, which are commonly used to measure the strength of ambiguity aversion in quantitative settings.

4.2 Age Profiles of Portfolio Allocations

Figure 6 plots the risky asset share for agents of different ages in the calibrated model, alongside the empirical age profile from the SCF. The model replicates the increasing age profile of risky asset shares, and the fact this profile is steeper at younger ages. This is despite the simplicity of the model, and the fact that this profile was not targeted at all in the calibration.

Figure 6: Model-generated vs. empirical risky asset shares in 2019.



Note: The solid line is constructed using the model, with calibration described in Section 4.1 and Appendix C.1. Each point is the mid-point of a ten-year age range. The dashed line plots the conditional risky asset share in the 2019 SCF. This is defined as the share of risky assets in total financial assets among households who participate in risky asset markets.

Of course, the empirical age-profile shown is a snapshot of a particular point in time, and so conflates age, period, and cohort effects. A large literature attempts to disentangle the pure age effect using estimated models and a range of identifying assumptions (see

[19]Note that in the solution for our object of interest $\alpha_{j,t}$ (equation 22), wealth and θ enter as the product $\theta w_{j,t}^{1-\gamma}$. Since θ is calibrated to match the average $\alpha_{j,t}$ in the data, the units of wealth used in the calibration become irrelevant for this variable. Replacing $w_{j,t}$ with $\Lambda w_{j,t}$ for some constant $\Lambda > 0$ would imply the calibrated θ would become $\Lambda^{\gamma-1}\theta$, leaving all variation in $\alpha_{j,t}$ unchanged.

Gomes and Smirnova, 2021, for a recent example). However, this is not our focus. A key part of our mechanism is that changes in life expectancy affect portfolio decisions, through changes in ambiguity-driven belief distortions. Since life expectancy changes over time and by cohort, we do not want to strip out these effects. Indeed, exploring how this profile might change over time is the purpose of Section 4.3.

In the model, the large initial rise in the risky asset share between the ages of 30 and 40 occurs largely because wealth increases substantially in this part of the population: the median wealth for people aged 40 is more than 2.5 times larger than those aged 30 in the SCF data. The wealth effect (Section 3.2.2) therefore implies that 40-year-olds are substantially more optimistic about risky asset returns than 30-year-olds, and invest more in risky assets as a result.

However, this rise in wealth is not the only reason for agents to become more optimistic as they age. Figure 7 shows how $\alpha_{j,t}$ would change over the agent's life cycle if (a) survival probabilities were constant for all age groups, or (b) wealth was constant for all age groups. In the first case, wealth varies as in the baseline calibration (Figure 6), but at each age survival probabilities are set as if the agent is still 30 years old. In the second, age and survival probabilities vary as in the baseline calibration, but wealth is held at that of the median 30-year-old. Since $\alpha_{j,t}$ is a non-linear combination of wealth and age effects, this is not a strict decomposition of the two channels. Rather, these alternative age profiles simply show the strength of the wealth and age effects in isolation. Although the wealth effect alone produces the same qualitative age profile of $\alpha_{j,t}$ as the full model, without age effects that profile is substantially less steep. Along with the increase in wealth, agents also therefore get more optimistic about risky asset returns because they get older.

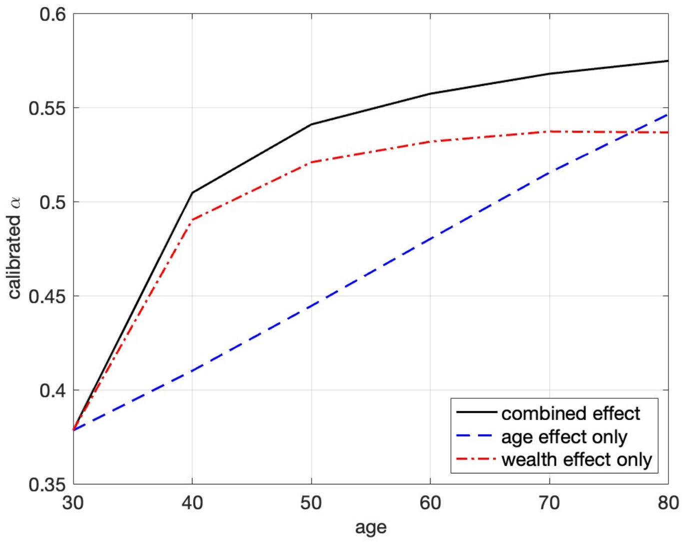
4.3 Projecting Asset Demand

We showed above that life expectancy is a key driver of the age profile of portfolio choices. As populations age, survival probabilities increase particularly for older cohorts. In our model, this leads to differential changes in the portfolio choices of different age groups, with implications for inequality within and between cohorts.

To explore this dependence on demographics, we take the calibrated model and replace the 2019 survival rates with demographic projections for the year 2100 in the US.[20] As well as ϕ_j, we also re-calibrate the maximum lifespan J using the same approach as before: we find the first age at which the annual mortality rate is expected to exceed 50%, which for 2100 implies $J = 117$. These projections come from the US Office of

[20] A related exercise is performed in Auclert et al. (2021), which takes an OLG model and forecasts future wealth-to-income ratios by holding fixed the average wealth and income of each age group, but varying the proportions of households of each age according to UN projections.

Figure 7: Model-generated risky asset shares in 2019, comparison to wealth and age effects in isolation.



Note: Plots constructed using the calibration and data described in Section 4.1 and Appendix C.1. Each point is the mid-point of a ten-year age range. To construct the 'wealth effect' line, the agent's age j is held fixed at 30, while $w_{j,t}$ is varied to match median financial assets for each age group. To construct the 'age effect' line, again the main calibration is used, except $w_{j,t}$ is set to the median wealth of a 30-year-old for agents of all ages j.

the Chief Actuary, who predict survival probabilities rising substantially, particularly for the oldest age groups.[21] This exercise therefore gives a projection of the direct effect of extending life expectancy on portfolio choices, holding everything else fixed.

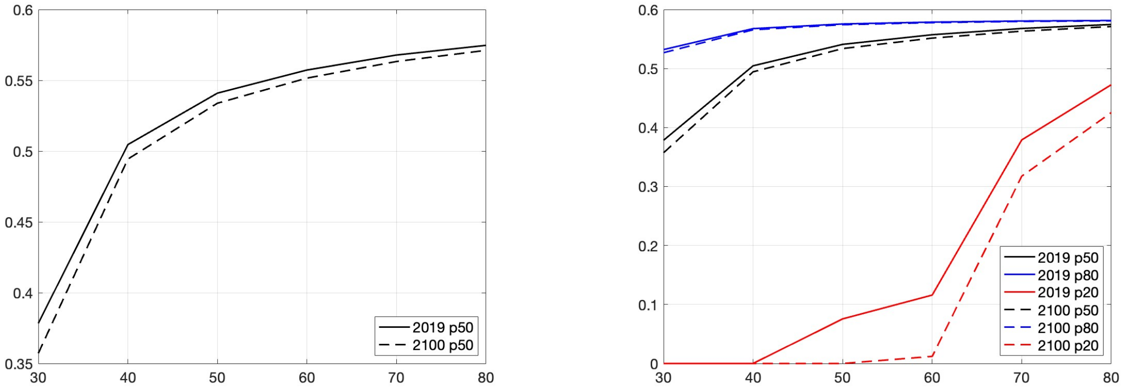
Figure 8a shows the results, plotting the model-implied age profile of risky asset shares in 2019 and 2100, for households at the median wealth for their age group (as in Figures 6 and 7). Solid lines plot $\alpha_{j,2019}$ and dashed lines plot the equivalent $\alpha_{j,2100}$. The increase in life expectancy by 2100 causes young households to invest less in risky assets, as they distort their beliefs more strongly towards low risky returns. For older households, return expectations decline by less, so they become more optimistic relative to the young.[22] The age profile of risky asset shares therefore becomes steeper, in line with the results with $J = 2$ (Corollary 2). Quantitatively, however, this effect is modest: the gap between the risky asset shares of those aged 80 and 30 rises from 19.6 p.p. to 21.4 p.p., an increase of 9%.

Plausible increases in survival probabilities are therefore not large enough to cause

[21] For example, in 2019 the death rate among 70-year-olds in the US was 1.9%. In 2100 that is projected to fall to 1.0%. The data is available at https://www.ssa.gov/oact/HistEst/Death/2023/DeathProbabilities2023.html

[22] Even 80 year-old households reduce $\alpha_{j,t}$ slightly in this simulation, though households above this age (not plotted) do increase $\alpha_{j,t}$, in line with the results in Section 3.

Figure 8: Model-implied risky asset shares, 2019 and 2100.



(a) Median wealth

(b) 20th, 50th, 80th wealth percentiles

Note: Plots constructed using the calibration and data described in Section 4.1 and Appendix C.1. Each point is the mid-point of a ten-year age range. To construct the '2100' line, the main calibration is used, except that ϕ_j is replaced with projected survival probabilities for 2100, computed as described in Appendix 4.1.

very large changes in the decisions of agents with median levels of wealth. However, this is not true for all agents. Figure 8b again plots the age profile of $\alpha_{j,t}$ in 2019 and 2100, but for agents at the 20th and 80th percentile of the wealth distribution in each age group. As with Figure 6 above, these percentiles of wealth are taken from the 2019 SCF.

The age profiles for less wealthy agents in particular are much more strongly affected by demographic change. Increases in survival rates mean they start to participate in the risky asset market later in life,[23] and the share invested in risky assets drops substantially at all ages with positive $\alpha_{j,t}$. These changes in life expectancy have stronger effects on poorer households for the reasons identified in Corollary 4: their marginal utility of wealth is more sensitive to changes in age. Through the same channel, increasing survival rates have very little effect on agents at the 80th percentile of the wealth distribution.

Importantly, note that this projection only captures changes directly due to the age effect on ambiguity aversion. If the wealth distribution or asset returns change over time, they would further alter these results. Most notably, we found in Section 3.3 that increased longevity causes older agents to become relatively wealthier through a number of channels. Since return expectations become more optimistic with wealth (Corollary 3), this would further amplify the changes in belief heterogeneity across ages projected here. In addition, if the equity premium continues its recent rising trend (as suggested in Section 3.5), the wealth of older agents with greater risky asset shares would grow even more rapidly relative to their younger counterparts.

[23] We implement the binding $\alpha_{j,t} \geq 0$ constraint with a simple piecewise-linear solution to the model. If the solution to the unconstrained model produces $\alpha_{j,t} < 0$, we replace it with $\alpha_{j,t=0}$.

5 Conclusion

We develop a model in which investors face ambiguity over expected returns on risky assets. In contrast to previous literature, we allow agents to choose the degree to which they respond to this ambiguity optimally. With the same preferences, ambiguity aversion causes stronger distortions to return expectations among agents whose utility is very sensitive to risky asset returns. This implies differential effects of ambiguity on expected returns and portfolio choices for agents with different levels of wealth, and different life expectancies.

In particular, as life expectancy rises, younger investors become more sensitive to rates of return, which means they distort their beliefs to be more pessimistic about the returns to risky assets, and they invest less in those assets. In contrast, in the empirically reasonable case where the elasticity of intertemporal substitution is less than 1, older agents become more optimistic about risky asset returns, and allocate a greater share of their savings to them. In this case wealthier households are also endogenously more optimistic about risky asset returns, fueling greater savings rates and risky asset shares, as documented in Straub (2019), Briggs et al. (2020) and others.

This model generates empirically plausible age profiles of risky asset shares in the US. Moreover, the mechanism suggests that as the population ages and life expectancies increase, older households will increase the share of their wealth invested in risky assets relative to the young. Such a shift would have important implications for inequality within and across generations. Further research could integrate these channels with other life-cycle mechanisms in richer quantitative models, and analyse potential policy responses.

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A Proofs

Proposition 1: Since $\phi_J = 0$ and there are no bequests, there is no incentive to save at age J. As we also impose that $w_{J+1} \geq 0$ (5), all agents at age J choose to consume all of their wealth. With no continuation value, the value function of these agents is:

$$V_J(w_J) = \frac{w_J^{1-\gamma}}{1-\gamma} \tag{54}$$

For all $j < J$, we now conjecture that the value function takes the same functional form:

$$V_j(w_j) = A_j \frac{w_j^{1-\gamma}}{1-\gamma} \tag{55}$$

for some age-dependent constant A_j. The expectation of the next-period value function becomes:

$$E_{j,t}[V_{j+1}(w_{j+1})] = E_{j,t}[A_{j+1}\frac{w_{j+1}^{1-\gamma}}{1-\gamma}] \tag{56}$$

$$\approx \frac{A_{j+1}}{1-\gamma}(w_j - c_{j,t})^{1-\gamma} E_{j,t}\left\{\exp\left[r^f + \alpha_{j,t}(r_{t+1} - r^f) + \frac{1}{2}\alpha_{j,t}(1-\alpha_{j,t})\sigma^2\right]\right\}^{1-\gamma} \tag{57}$$

where equation (57) uses the log-linear approximation to the one-period excess return around zero in equation (7) and Campbell (1993). Since r_{t+1} is normally distributed, we can further write:

$$E_{j,t}[V_{j+1}(w_{j+1})] = \frac{A_{j+1}}{1-\gamma}(w_j - c_{j,t})^{1-\gamma} \exp\left\{(1-\gamma)\left[r^f + \alpha_{j,t}(\tilde{\mu} - r^f) + \frac{1}{2}\alpha_{j,t}(1-\alpha_{j,t})\sigma^2\right] + \frac{1}{2}(1-\gamma)^2\alpha_{j,t}^2\sigma^2\right\} \tag{58}$$

Using this in equation (3) and taking first order conditions with respect to $\alpha_{j,t}, c_{j,t}$ gives equations (12) and (13), with b_{j+1} as in equation (11) (noting that $\mu \equiv \tilde{\mu} + \sigma^2/2$).

Finally, we verify the conjecture in equation (55) for $j = 0, ..., J-1$. Substituting equation (58) into the value function (3) we have:

$$A_j\frac{w_j^{1-\gamma}}{1-\gamma} = \frac{(c_{j,t}^*)^{1-\gamma}}{1-\gamma} + \beta\phi_j\frac{A_{j+1}}{1-\gamma}(w_j - c_{j,t}^*)^{1-\gamma}\left\{1 + (1-\gamma)[r^f + (\mu - r^f)\alpha_{j,t}^*] - \frac{1}{2}\gamma(1-\gamma)\sigma^2(\alpha^*)^2\right\} \tag{59}$$

where $\alpha^*, c_{j,t}^*$ denote the optimal risky share and consumption from equations (12) and (13) respectively. Since α^* is a constant, and $c_{j,t}^*$ is proportional to w_j, this verifies the conjectured functional form for $V_j(w_j)$ (55). Matching coefficients, we obtain equation (10).

Lemma 1: First conjecture that the value function takes the form:

$$V_j^\theta(w_j) = A_j\frac{w_j^{1-\gamma}}{1-\gamma} + \theta B_j\frac{w_j^{2(1-\gamma)}}{2(1-\gamma)} + O(\theta^2) \tag{60}$$

Taking the same log-linear approximation approach as in (7) to the distorted returns in

(14), we can write

$$
\begin{aligned}
E_{j,t}[V_{j+1}^{\theta}(w_{j+1})] &\approx \frac{A_{j+1}}{1-\gamma}(w_j - c_{j,t})^{1-\gamma}\{1 + (1-\gamma)[r^f + (\widetilde{\mu} - r^f + \sigma\nu_{j,t})\alpha_{j,t}] \\
&\quad + \frac{1}{2}(1-\gamma)\sigma^2(\alpha_{j,t} - \gamma\alpha_{j,t}^2)\} \\
&\quad + \theta\frac{B_{j+1}}{2(1-\gamma)}(w_j - c_{j,t})^{2(1-\gamma)}\{1 + 2(1-\gamma)[r^f + (\widetilde{\mu} - r^f + \sigma\nu_{j,t})\alpha_{j,t}] \\
&\quad + (1-\gamma)\sigma^2(\alpha_{j,t} + \alpha_{j,t}^2 - 2\gamma\alpha_{j,t}^2)\} \\
&= E[V_{j+1}(w_{j+1}^*)] + \frac{A_{j+1}}{1-\gamma}(w_j - c_{j,t})^{1-\gamma}(1-\gamma)\sigma\alpha_{j,t}\nu_{j,t} \\
&\quad + \theta\frac{B_{j+1}}{2(1-\gamma)}(w_j - c_{j,t})^{2(1-\gamma)}2(1-\gamma)\sigma\alpha_{j,t}\nu_{j,t} \qquad (61)
\end{aligned}
$$

where in the first approximation we drop the term with θ^2 and higher orders, and use the approximation:

$$R_{t+1} + \sigma_1\nu_{j,t} \approx \exp(r_{t+1} + \sigma\nu_{j,t}) \qquad (62)$$

In equation (62), σ_1 is the standard deviation of R_{t+1}, which equals to $[\exp(\sigma^2) - 1]^{\frac{1}{2}}[\exp(2\widetilde{\mu} + \sigma^2)]^{\frac{1}{2}}$. Note that $\theta\nu_{j,t} << \nu_{j,t}$, we can further approximate $E_{j,t}[V_{j+1}^{\theta}(w_{j+1})]$ as:

$$E_{j,t}[V_{j+1}^{\theta}(w_{j+1})] \approx E_{j,t}[V_{j+1}(w_{j+1}^*)] + \frac{A_{j+1}}{1-\gamma}(w_j - c_{j,t})^{1-\gamma}(1-\gamma)\sigma\alpha_{j,t}\nu_{j,t} \qquad (63)$$

Proposition 2: Continue with the guess of the value function's form:

$$
\begin{aligned}
V_j^{\theta}(w_j) &= A_j\frac{w_j^{1-\gamma}}{1-\gamma} + \theta B_j\frac{w_j^{2(1-\gamma)}}{2(1-\gamma)} + O(\theta^2) \qquad (64) \\
&\approx V_j^0(w_j) + \theta V_j^1(w_j) \qquad (65)
\end{aligned}
$$

Note that A_j is the same as in the benchmark model. When there is no ambiguity aversion (i.e. $\theta = 0$), the value function degenerates into that of the benchmark model.

It is intuitive to conjecture that the optimal portfolio choice and consumption take the following forms:

$$
\begin{aligned}
\alpha_{j,t} &= \alpha_{j,t}^* + \theta\alpha_{j,t}' \qquad (66) \\
c_{j,t} &= c_{j,t}^* + \theta c_{j,t}' \qquad (67)
\end{aligned}
$$

where $\alpha^*, c_{j,t}^*$ are the solutions without ambiguity defined in Proposition 1.

Then we can approximate the value function by dropping the term with θ^2 and higher orders:

$$
\begin{aligned}
V_j^\theta(w_j) &= \max_{c,\alpha}\{\frac{c_{j,t}^{1-\gamma}}{1-\gamma} + \beta\phi_j[-\frac{1}{2}\theta A_{j+1}^2(w_j - c_{j,t})^{2-2\gamma}\sigma^2\alpha_{j,t}^2 + E_{j,t}[V_{j+1}^\theta(w_{j+1}^*)]]\} \\
&\approx \max_{c,\alpha}\{\frac{c_{j,t}^{1-\gamma}}{1-\gamma} + \beta\phi_j[-\frac{1}{2}\theta A_{j+1}^2(w_j - c_{j,t})^{2-2\gamma}\sigma^2\alpha_{j,t}^2 + E_{j,t}[V_{j+1}^0(w_{j+1}^*)] \\
&\quad +\theta E_{j,t}[V_{j+1}^1(w_{j+1}^*)]]\}
\end{aligned}
$$

where using log linearization we have:

$$
\begin{aligned}
E_{j,t}[V_{j+1}^0(w_{j+1}^*)] &\approx \frac{A_{j+1}}{1-\gamma}(w_j - c_{j,t})^{1-\gamma}\{1 + (1-\gamma)[r^f + (\widetilde{\mu} - r^f)\alpha_{j,t}] \\
&\quad +\frac{1}{2}(1-\gamma)\sigma^2(\alpha_{j,t} - \gamma\alpha_{j,t}^2)\} \\
E_{j,t}[V_{j+1}^1(w_{j+1}^*)] &\approx \frac{B_{j+1}}{2(1-\gamma)}(w_j - c_{j,t})^{2(1-\gamma)}\{1 + 2(1-\gamma)[r^f + (\widetilde{\mu} - r^f)\alpha_{j,t}] \\
&\quad +(1-\gamma)\sigma^2(\alpha_{j,t} + \alpha_{j,t}^2 - 2\gamma\alpha_{j,t}^2)\}
\end{aligned}
$$

The FOC w.r.t. $\alpha_{j,t}$ gives:

$$
\begin{aligned}
0 &= -\theta A_{j+1}^2(w_j - c_{j,t})^{1-\gamma}\sigma^2\alpha_{j,t} + A_{j+1}\{(\widetilde{\mu} - r^f) + \frac{1}{2}\sigma^2 - \gamma\sigma^2\alpha_{j,t}\} \\
&\quad +\theta B_{j+1}(w_j - c_{j,t})^{1-\gamma}\{(\widetilde{\mu} - r^f) + \frac{1}{2}\sigma^2 + (1-2\gamma)\sigma^2\alpha_{j,t}\}
\end{aligned}
$$

Substituting $\mu = \widetilde{\mu} + \frac{1}{2}\sigma^2$ into the equation gives:

$$
\begin{aligned}
0 &= -\theta A_{j+1}^2(w_j - c_{j,t})^{1-\gamma}\sigma^2\alpha_{j,t} + A_{j+1}\{(\mu - r^f) - \gamma\sigma^2\alpha_{j,t}\} \\
&\quad +\theta B_{j+1}(w_j - c_{j,t})^{1-\gamma}\{(\mu - r^f) + (1-2\gamma)\sigma^2\alpha_{j,t}\}
\end{aligned}
$$

By plugging $\alpha_{j,t} = \alpha_{j,t}^* + \theta\alpha_{j,t}' = \frac{\mu - r}{\gamma\sigma^2} + \theta\alpha_{j,t}'$, we get:

$$
\begin{aligned}
0 &= -\theta A_{j+1}^2(w_j - c_{j,t})^{1-\gamma}\sigma^2(\alpha_{j,t}^* + \theta\alpha_{j,t}') - \theta A_{j+1}\gamma\sigma^2\alpha_{j,t}' \\
&\quad +\theta B_{j+1}(w_j - c_{j,t})^{1-\gamma}\{(\mu - r^f) + (1-2\gamma)\sigma^2(\frac{\mu - r^f}{\gamma\sigma^2} + \theta\alpha_{j,t}')\}
\end{aligned}
$$

Further approximation by dropping terms with θ^2 and simplification give:

$$
\begin{aligned}
0 &\approx -A_{j+1}^2(w_j - c_{j,t})^{1-\gamma}\sigma^2\alpha_{j,t}^* - A_{j+1}\gamma\sigma^2\alpha'_{j,t} \\
&\quad + B_{j+1}(w_j - c_{j,t})^{1-\gamma}(\mu - r^f)(\frac{1}{\gamma} - 1) \\
\alpha'_{j,t} &= -\frac{A_{j+1}^2 + B_{j+1}(\gamma - 1)}{A_{j+1}\gamma}\alpha_{j,t}^*(w_j - c_{j,t}^*)^{1-\gamma} \\
&= -\frac{A_{j+1}^2 + B_{j+1}(\gamma - 1)}{A_{j+1}\gamma}\alpha_{j,t}^*(\frac{w_j}{1 + b_{j+1}})^{1-\gamma} \qquad (68)
\end{aligned}
$$

Then we can simplify $E_{j,t}[V_{j+1}^0(w_{j+1}^*)]$ and $E_{j,t}[V_{j+1}^1(w_{j+1}^*)]$:

$$
\begin{aligned}
E_{j,t}[V_{j+1}^0(w_{j+1}^*)] &\approx \frac{A_{j+1}}{1-\gamma}(w_j - c_{j,t})^{1-\gamma}\{1 + (1-\gamma)r^f + \frac{1}{2}\gamma(1-\gamma)\sigma^2(\alpha_{j,t}^*)^2\} \\
E_{j,t}[V_{j+1}^1(w_{j+1}^*)] &\approx \frac{B_{j+1}}{2(1-\gamma)}(w_j - c_{j,t})^{2(1-\gamma)}\{1 + 2(1-\gamma)r^f \\
&\quad + (1-\gamma)\sigma^2(\alpha_{j,t}^*)^2\}
\end{aligned}
$$

We move onto the RHS of the Bellman equation:

$$
RHS = \frac{c_{j,t}^{1-\gamma}}{1-\gamma} + \beta\phi_j[-\frac{1}{2}\theta A_{j+1}^2(w_j - c_{j,t})^{2-2\gamma}\sigma^2(\alpha_{j,t}^*)^2 + E_t[V_{j+1}^0(w_{j+1}^*)] + \theta E_{j,t}[V_{j+1}^1(w_{j+1}^*)]]
$$

The FOC w.r.t $c_{j,t}$ gives:

$$
\begin{aligned}
0 &= c_{j,t}^{-\gamma} - (w_j - c_{j,t})^{-\gamma}b_{j+1}^{-\gamma} \\
&\quad + \theta\beta\phi_j(w_j - c_{j,t})^{1-2\gamma}\{(A_{j+1}^2 - B_{j+1})(1-\gamma)\sigma^2(\alpha_{j,t}^*)^2 - B_{j+1}[1 + 2(1-\gamma)r^f]\}
\end{aligned}
$$

By plugging $c_{j,t} = c_{j,t}^* + \theta c'_{j,t}$, we get:

$$
\gamma b_{j+1}^{-\gamma-1}c'_{j,t} + \gamma b_{j+1}^{-\gamma}c'_{j,t} = \beta\phi_j(w_j - c_{j,t}^*)^{2-\gamma}[(A_{j+1}^2 - B_{j+1})(1-\gamma)\sigma^2(\alpha_{j,t}^*)^2 - B_{j+1}[1 + 2(1-\gamma)r^f]]
$$

Therefore:

$$
\begin{aligned}
c'_{j,t} &= \frac{\beta\phi_j\{(A_{j+1}^2 - B_{j+1})(1-\gamma)\sigma^2(\alpha_{j,t}^*)^2 - B_{j+1}[1 + 2(1-\gamma)r^f]\}}{\gamma b_{j+1}^{-\gamma}(1 + b_{j+1}^{-1})}(w_j - c_{j,t}^*)^{2-\gamma} \\
&= \frac{\beta\phi_j\{(A_{j+1}^2 - B_{j+1})(1-\gamma)\sigma^2(\alpha_{j,t}^*)^2 - B_{j+1}[1 + 2(1-\gamma)r^f]\}}{\gamma b_{j+1}^{-\gamma}(1 + b_{j+1}^{-1})}(\frac{w_j}{1 + b_{j+1}})^{2-\gamma} \qquad (69)
\end{aligned}
$$

where we use first-order Taylor approximation:

$$\begin{aligned} c_{j,t}^{-\gamma} &\approx (c_{j,t}^*)^{-\gamma} - \gamma (c_{j,t}^*)^{-\gamma-1}\theta c_{j,t}' \\ (w_j - c_{j,t})^{-\gamma} &\approx (w_j - c_{j,t}^*)^{-\gamma} + \gamma (w_j - c_{j,t}^*)^{-\gamma-1}\theta c_{j,t}' \end{aligned}$$

Plugging the above results into the Bellman equation gives:

$$\begin{aligned} B_j \frac{w_j^{2(1-\gamma)}}{2(1-\gamma)} &= [\frac{1}{\gamma(1+b_{j+1}^{-1})} - \frac{1}{2(1-\gamma)}]\beta\phi_j\{(A_{j+1}^2 - B_{j+1})(1-\gamma)\sigma^2(\alpha_{j,t}^*)^2 \\ &\quad -B_{j+1}[1+2(1-\gamma)r^f]\}(w_j - c_{j,t}^*)^{2-2\gamma} \end{aligned}$$

where we use first-order Taylor approximation again:

$$c_{j,t}^{1-\gamma} \approx (c_{j,t}^*)^{1-\gamma} - (\gamma-1)(c_{j,t}^*)^{-\gamma}\theta c_{j,t}'$$

Matching coefficients w.r.t. w_j gives:

$$B_j = \beta\phi_j[\frac{2(1-\gamma)}{\gamma(1+b_{j+1}^{-1})} - 1]\{(A_{j+1}^2 - B_{j+1})(1-\gamma)\sigma^2(\alpha_{j,t}^*)^2 - B_{j+1}[1+2(1-\gamma)r^f]\}(\frac{1}{1+b_{j+1}})^{2-2\gamma} \tag{70}$$

with $B_J = 0$ because the agents in the last period do not invest and hence are not exposed to uncertainty. The system can be solved backward.

We can express $\alpha_{j,t}$ and $c_{j,t}$ in another way:

$$\alpha_{j,t} = \alpha^* + \theta\alpha^* w_j^{1-\gamma}\Omega_{\alpha j} \tag{71}$$

$$c_{j,t} = c_{j,t}^* + \theta w_j^{2-\gamma}\Omega_{cj} \tag{72}$$

where

$$\Omega_{\alpha j} = -\frac{A_{j+1}^2 + B_{j+1}(\gamma-1)}{\gamma A_{j+1}(1+b_{j+1})^{\gamma-1}} \tag{73}$$

$$\Omega_{cj} = \frac{\beta\phi_j\{(A_{j+1}^2 - B_{j+1})(1-\gamma)\sigma^2(\alpha^*)^2 - B_{j+1}[1+2(1-\gamma)r^f]\}}{\gamma b_{j+1}^{-\gamma-1}(1+b_{j+1})^{3-\gamma}} \tag{74}$$

$$sgn(\frac{\partial\Omega_{c1}}{\partial\phi_1}) = sgn((1-\gamma)\frac{\partial\phi_1^{-\frac{1}{\gamma}}(1+\tilde{b}\phi_1^{-\frac{1}{\gamma}})^{\gamma-3}}{\partial\phi_1}) \tag{75}$$

$$= sgn((\gamma-1)(1+(\tilde{b}+\gamma-3)\phi_1^{-\frac{1}{\gamma}})) \tag{76}$$

Proposition 3: The result follows largely from applying Proposition 1.

Since $J = 2$, we have that $A_2 = 1$, and so:

$$b_2 = \left[\beta\phi_1[1 + (1-\gamma)r^f + \frac{1}{2}(1-\gamma)\frac{(\mu - r^f)^2}{\gamma\sigma^2}]\right]^{-\frac{1}{\gamma}} \tag{77}$$

$$= \phi_1^{-\frac{1}{\gamma}}\tilde{b} \tag{78}$$

where $\tilde{b}$ is defined in (26).

Applying the definitions in Proposition 1, we further obtain:

$$A_1 = \left(\frac{\phi_1^{\frac{1}{\gamma}} + \tilde{b}}{\tilde{b}}\right)^{\gamma} \tag{79}$$

$$b_1 = \frac{\tilde{b}}{\phi_1^{\frac{1}{\gamma}} + \tilde{b}} \tag{80}$$

Substituting these into equation (13) gives equations (24) and (25). Finally, note using equations (5) and (4) that:

$$c_{2,t+2} = w_2 = w_0 R^p_{0,t+1} R^p_{1,t+2} - c_{0,t} R^p_{0,t+1} R^p_{1,t+2} - c_{1,t+1} R^p_{1,t+2} \tag{81}$$

Substituting in equations (24) and (25) implies equation (27).

Proposition 4: Since in a simple three-period model, the old do not invest in the stock market, we have $\nu_{2,t} = 0$. Substituting equations (22) and (23) into equation (19), and using the expressions for A_1, b_1 and b_2 gives us the optimal distortions for the young and the middle-aged.

Corollary 1: Using Proposition 4 we have that:

$$\nu_{1,t}(1) = -\frac{\theta\sigma\alpha^*}{(1+\tilde{b})^{1-\gamma}} w_1^{1-\gamma} \tag{82}$$

And:

$$\nu_{0,t}(0) = -\frac{\theta\sigma\alpha^*}{\tilde{b}^{\gamma-1}(\tilde{b} + \tilde{b}^2)^{1-\gamma}} w_0^{1-\gamma} = -\frac{\theta\sigma\alpha^*}{(1+\tilde{b})^{1-\gamma}} w_0^{1-\gamma} \tag{83}$$

Therefore $\nu_{0,t}(0) = \nu_{1,t}(1)$

Corollary 2: For the middle-aged:

$$\frac{\partial \nu_{1,t}}{\partial \phi_1} = -\theta\sigma\alpha^*_{1,t}(w_1)^{1-\gamma}\frac{\partial(1+\phi_1^{-\frac{1}{\gamma}}\tilde{b})^{\gamma-1}}{\partial \phi}$$

$$= \frac{\gamma-1}{\gamma}\theta\sigma\alpha^*_1(w_1)^{1-\gamma}(1+\phi_1^{-\frac{1}{\gamma}}\tilde{b})^{\gamma-2}\phi_1^{-\frac{1}{\gamma}-1}\tilde{b}\begin{cases} <0 & \text{if } \gamma<1 \\ =0 & \text{if } \gamma=1 \\ >0 & \text{if } \gamma>1 \end{cases} \quad (84)$$

For the young:

$$\frac{\partial \nu_{0,t}}{\partial \phi_1} = -\theta\sigma\alpha^*_{0,t}(w_0)^{1-\gamma}\tilde{b}^{-\gamma}(\phi_1^{\frac{1}{\gamma}}+\tilde{b}+\tilde{b}^2)^{-2+\gamma}\frac{1}{\gamma}\phi_1^{\frac{1}{\gamma}-1}(\tilde{b}^2+\gamma\phi_1^{\frac{1}{\gamma}}+\tilde{b}\gamma)<0 \quad (85)$$

Corollary 3: For the middle-aged:

$$\frac{\partial \nu_{1,t}}{\partial w_1} = -\frac{\theta\sigma\alpha^*\phi_1^{\frac{1-\gamma}{\gamma}}}{(\phi_1^{\frac{1}{\gamma}}+\tilde{b})^{1-\gamma}}w_1^{-\gamma}(1-\gamma)\begin{cases} <0 & \text{if } \gamma<1 \\ =0 & \text{if } \gamma=1 \\ >0 & \text{if } \gamma>1 \end{cases}$$

For the young:

$$\frac{\partial \nu_{0,t}}{\partial w_0} = -\frac{\theta\sigma\alpha^*(\phi_1^{\frac{1}{\gamma}}+\tilde{b})}{\tilde{b}^{\gamma}(\phi_1^{\frac{1}{\gamma}}+\tilde{b}+\tilde{b}^2)^{1-\gamma}}w_0^{-\gamma}(1-\gamma)\begin{cases} <0 & \text{if } \gamma<1 \\ =0 & \text{if } \gamma=1 \\ >0 & \text{if } \gamma>1 \end{cases}$$

Corollary 4: Differentiating equations 84 and 85 with respect to w_0, w_1 respectively:

$$\frac{\partial}{\partial w_0}\left(\frac{\partial \nu_0}{\partial \phi_1}\right) = -\frac{(1-\gamma)}{\gamma}\theta\sigma\alpha^*_{0,t}(w_0)^{-\gamma}\tilde{b}^{-\gamma}(\phi_1^{\frac{1}{\gamma}}+\tilde{b}+\tilde{b}^2)^{-2+\gamma}\phi_1^{\frac{1}{\gamma}-1}(\tilde{b}^2+\gamma\phi_1^{\frac{1}{\gamma}}+\tilde{b}\gamma)$$

$$\frac{\partial}{\partial w_1}\left(\frac{d\nu_1}{d\phi_1}\right) = -\frac{(1-\gamma)^2}{\gamma}\theta\sigma\alpha^*_1(w_1)^{-\gamma}(1+\phi_1^{-\frac{1}{\gamma}}\tilde{b})^{\gamma-2}\phi_1^{-\frac{1}{\gamma}-1}\tilde{b}$$

If $\gamma = 0$, both differentials equal 0. If $\gamma \neq 0$, then:

$$\begin{aligned}\frac{\partial}{\partial w_0}\left(\frac{\partial \nu_0}{\partial \phi_1}\right) &\propto -(1-\gamma) \\ \frac{\partial}{\partial w_1}\left(\frac{\partial \nu_1}{\partial \phi_1}\right) &\propto -(1-\gamma)^2\end{aligned}$$

Combined with the signs derived in Corollary 2 this delivers Corollary 4.

Lemma 2: To reduce notation, in this proof we use AD_r to denote the relative aggregate demand for risky assets:

$$AD_r \equiv \frac{AD(risky)}{AD(safe) + AD(risky)} \tag{86}$$

Using equation 12 to substitute out for α^*, equation 48 can be rewritten:

$$AD_r = \frac{\mu - r^f}{\gamma\sigma^2}\left(1 + \theta(\Gamma_0 + \Gamma_1)\right) \tag{87}$$

where:

$$\Gamma_j = \frac{\Omega_{\alpha,j} w_j^{1-\gamma}(w_j - c_j)}{w_0 - c_0 + w_1 - c_1} \tag{88}$$

for $j \in \{0, 1\}$.

From this we have:

$$\frac{\partial AD_r}{\partial \mu} = \frac{1}{\gamma\sigma^2}\left(1 + \theta(\Gamma_0 + \Gamma_1)\right) + \frac{\mu - r^f}{\gamma\sigma^2}\theta\left(\frac{\partial \Gamma_0}{\partial \mu} + \frac{\partial \Gamma_1}{\partial \mu}\right) \tag{89}$$

This derivative is strictly positive if:

$$-\theta\left(\frac{\partial \Gamma_0}{\partial \mu} + \frac{\partial \Gamma_1}{\partial \mu}\right) < AD_r \frac{\gamma\sigma^2}{(\mu - r^f)^2} \tag{90}$$

Similarly, differentiating equation 87 with respect to r^f yields:

$$\frac{\partial AD_r}{\partial r^f} = -\frac{1}{\gamma\sigma^2}\left(1 + \theta(\Gamma_0 + \Gamma_1)\right) + \frac{\mu - r^f}{\gamma\sigma^2}\theta\left(\frac{\partial \Gamma_0}{\partial r^f} + \frac{\partial \Gamma_1}{\partial r^f}\right) \tag{91}$$

This derivative is strictly negative if:

$$\theta \left(\frac{\partial \Gamma_0}{\partial r^f} + \frac{\partial \Gamma_1}{\partial r^f} \right) < AD_r \frac{\gamma \sigma^2}{(\mu - r^f)^2} \tag{92}$$

The right hand side is the same in conditions 90 and 92, and it is strictly positive. Let us start with condition 90. It is trivially satisfied if:

$$\left(\frac{\partial \Gamma_0}{\partial \mu} + \frac{\partial \Gamma_1}{\partial \mu} \right) \geq 0 \tag{93}$$

Outside of this case, condition 90 is satisfied if:

$$\theta < AD_r \frac{\gamma \sigma^2}{(\mu - r^f)^2} \left(-\frac{\partial \Gamma_0}{\partial \mu} - \frac{\partial \Gamma_1}{\partial \mu} \right)^{-1} \tag{94}$$

Now consider condition 92. Again, this is trivially satisfied if:

$$\left(\frac{\partial \Gamma_0}{\partial r^f} + \frac{\partial \Gamma_1}{\partial r^f} \right) \leq 0 \tag{95}$$

Outside of this case, condition 92 is satisfied if:

$$\theta < AD_r \frac{\gamma \sigma^2}{(\mu - r^f)^2} \left(\frac{\partial \Gamma_0}{\partial r^f} + \frac{\partial \Gamma_1}{\partial r^f} \right)^{-1} \tag{96}$$

A sufficient condition for both 90 and 92 to be satisfied is therefore:

$$\theta < \theta^* = AD_r \frac{\gamma \sigma^2}{(\mu - r^f)^2} \cdot \min \left(\left(-\frac{\partial \Gamma_0}{\partial \mu} - \frac{\partial \Gamma_1}{\partial \mu} \right)^{-1}, \left(\frac{\partial \Gamma_0}{\partial r^f} + \frac{\partial \Gamma_1}{\partial r^f} \right)^{-1} \right) \tag{97}$$

B Consumption and Saving with Ambiguity Aversion

In the benchmark without ambiguity, Proposition 3 implies that young and middle-aged agents consume less, and save more, when the probability of surviving to old age increases:

$$\frac{dc_j^*}{d\phi_1} < 0 \tag{98}$$

With ambiguity, we start with equation 23 for $j = \{0, 1\}$, and substitute out for c_j^*

using Proposition 3:

$$c_0 = \frac{\tilde{b}^2}{\phi_1^{\frac{1}{\gamma}} + \tilde{b} + \tilde{b}^2} w_0 + \theta w_0^{2-\gamma} \Omega_{c,0} \tag{99}$$

$$c_1 = \frac{\tilde{b}}{\phi_1^{\frac{1}{\gamma}} + \tilde{b} + \tilde{b}^2} R_0^p w_0 + \theta w_1^{2-\gamma} \Omega_{c,1} \tag{100}$$

Differentiating with respect to ϕ_1, we obtain:

$$\frac{dc_0}{d\phi_1} = -\frac{\tilde{b}^2 \phi_1^{\frac{1-\gamma}{\gamma}}}{\gamma(\phi_1^{\frac{1}{\gamma}} + \tilde{b} + \tilde{b}^2)} w_0 + \theta w_0^{2-\gamma} \frac{d\Omega_{c,0}}{d\phi_1} \tag{101}$$

$$\frac{dc_1}{d\phi_1} = -\frac{\tilde{b} \phi_1^{\frac{1-\gamma}{\gamma}}}{\gamma(\phi_1^{\frac{1}{\gamma}} + \tilde{b} + \tilde{b}^2)} R_0^p w_0 + \theta \left((2-\gamma) w_1^{1-\gamma} \Omega_{c,1} \frac{dw_1}{d\phi_1} + w_1^{2-\gamma} \frac{d\Omega_{c,1}}{d\phi_1} \right) \tag{102}$$

These derivatives are negative whenever:

$$\theta \frac{d\Omega_{c,0}}{d\phi_1} < \frac{\tilde{b}^2 \phi_1^{\frac{1-\gamma}{\gamma}}}{\gamma(\phi_1^{\frac{1}{\gamma}} + \tilde{b} + \tilde{b}^2)} w_0^{\gamma-1} \tag{103}$$

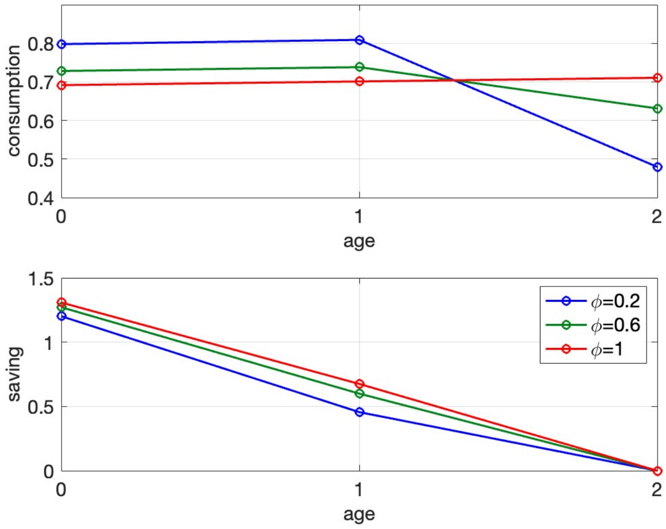
$$\theta \left((2-\gamma) \Omega_{c,1} \frac{dw_1}{d\phi_1} + w_1 \frac{d\Omega_{c,1}}{d\phi_1} \right) < \frac{\tilde{b} \phi_1^{\frac{1-\gamma}{\gamma}}}{\gamma(\phi_1^{\frac{1}{\gamma}} + \tilde{b} + \tilde{b}^2)} R_0^p w_0 w_1^{\gamma-1} \tag{104}$$

In both of these inequalities, the right hand side is strictly positive. Since $\Omega_{c,j}$ is independent of θ, there is therefore a θ^+ such that $\theta < \theta^+$ is sufficient to ensure both inequalities hold, and consumption of young and middle-aged agents falls as ϕ_1 rises.

Figure 9 shows a numerical example of this. It plots the consumption and saving paths for the ambiguity-averse agents with $J = 2$ studied in Section 3.2, with the same parameters as those in Figures 2 and 3.

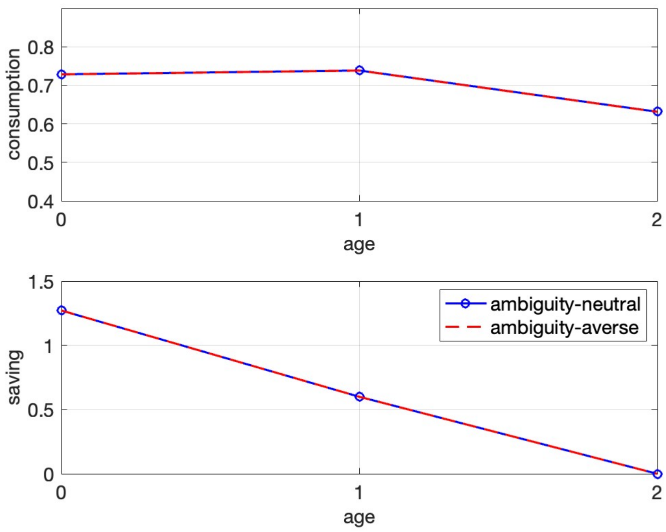
These paths are very similar to those without ambiguity aversion in Figure 1. Figure 10 plots the paths of consumption for $\phi_1 = 0.6$, with and without ambiguity.

Figure 9: Consumption and saving paths with ambiguity.



Note: Plots constructed using $J = 2$, $\mu = 0.06$, $r^f = 0.045$, $\sigma = 0.1$, $\theta = 0.045$, $\gamma = 3$, $\beta = 0.99$, $\phi_1 \in (0, 1]$, $w_0 = w_1 = 2$, and risky asset returns set to their expected level every period. This therefore abstracts from the effect of return shocks.

Figure 10: Consumption and saving paths with and without ambiguity, for $\phi_1 = 0.6$.



Note: Plots constructed using $J = 2$, $\mu = 0.06$, $r^f = 0.045$, $\sigma = 0.1$, $\phi_1 = 0.6$, $\gamma = 3$, $\beta = 0.99$, $\phi_1 \in (0, 1]$, $w_0 = w_1 = 2$, and risky asset returns set to their expected level every period. This therefore abstracts from the effect of return shocks.

C Quantitative Model Details

C.1 Data Construction and Calibration

SCF: Using the 2019 SCF microdata, we first group respondents into age categories: $< 35, 35 - 44, 45 - 54, 55 - 64, 65 - 74, 75+$. We then construct weighted percentiles of the distribution of financial wealth for each age group. The wealth variable used is total financial assets. When calibrating the model, for Figure 6 all agents are given the median wealth for their age group, in 1000s of dollars. In Figure 8, we solve the model assuming agents have the median wealth for their age, or other percentiles of the wealth distribution, depending on the line to be plotted.

The share of portfolios in risky assets used to calibrate θ, and plotted in Figure 6, is defined as stock holdings as share of family group's financial assets, among those who participate in the stock market.

Mortality: For the main calibration, we take mortality rates by age group from the NVSS, and compute ϕ_j as $1 - \text{mortality rate}_j$. For projections to 2100 in Section 4.3, we use projected mortality rates from the Office of the Chief Actuary. They project mortality rates for males and females separately, at every year of age up to 119. To compute projected aggregate survival rates from this, we first take data on sex ratios in the US in 2021 from the UN World Population Prospects 2022. This data is reported at selected ages,[24] so for the ages missing from the sex ratio data we assume the sex ratio is equal to that of the nearest cohort for which there is data. We then generate projected sex ratios at each age for all years up to 2100, by combining the mortality rates from the Office of the Chief Actuary for each sex, age, and year, with the assumption that the sex rate at birth will remain the same as it was in 2021. These projections are important, as in 2019 the female-male ratio rose sharply at older age groups due to women having longer life expectancy in the US. However, this life expectancy gap is projected to narrow in coming decades, which will reduce the female-male ratio in older age groups. Finally, we use the projected sex ratios to combine female and male mortality rates to give an aggregate mortality rate. 1- this mortality rate for each age group then gives projected ϕ_j in 2100.

C.2 Detection Error Probabilities

As is common in the literature on ambiguity aversion, we use model detection error probabilities (DEP) to infer whether agents are hedging against the models that are empirically plausible to generate the data we observe. Intuitively, we treat agents as

[24] These ages are 15, 20, 30, 40, 50, 60, 70, 80, 90, 100.

statisticians using likelihood ratio test to discriminate among models. DEP measures how far the alternative models can deviate from the approximating one without being discarded. Low values of DEP means that agents are unwilling to discard very different alternative models, which could be easily discriminated given observed data.

DEP assigns equal initial priors to the approximating and distorted models, hence it is the average of the probabilities of Type I and Type II errors. A Type I error occurs when the likelihood ratio test chooses the distorting model when the approximating model is the true data generating process. A Type II error is the reverse. Formally, DEP is defined as:

$$DEP = \frac{1}{2}Prob(\ln(\frac{L_A}{L_B}) < 0|A) + \frac{1}{2}Prob(\ln(\frac{L_B}{L_A}) > 0|B) \tag{105}$$

where A denotes the approximating model and B is the distorting model.

As discussed by Anderson et al. (2003), the following bound on the average error in using a likelihood ratio test to discriminate between the approximating and distorted models is useful when the data is of a continuous record with length T. The DEP bound in this discrete-time model can be approximated in the following way:

$$avg\ DEP \leq \frac{1}{2}E\ exp\{-\frac{1}{8}\int_0^T \nu^2(w_t)\,dt\} \tag{106}$$

$$\approx \frac{1}{2}E\ exp\{-\frac{1}{8}\sum_{t=0}^{T} \nu^2(w_t)\} \tag{107}$$

In the calibration, we take T=109 (consistent with the maximum lifespan in our calibration) and agents start investment at age 30. Here we plot how DEP changes with the level of ambiguity aversion. With the calibrated value of $\theta = 0.008$, we obtain a large DEP value. This implies that based on the observed data, it is not easy to distinguish the distorted model from the approximating model.

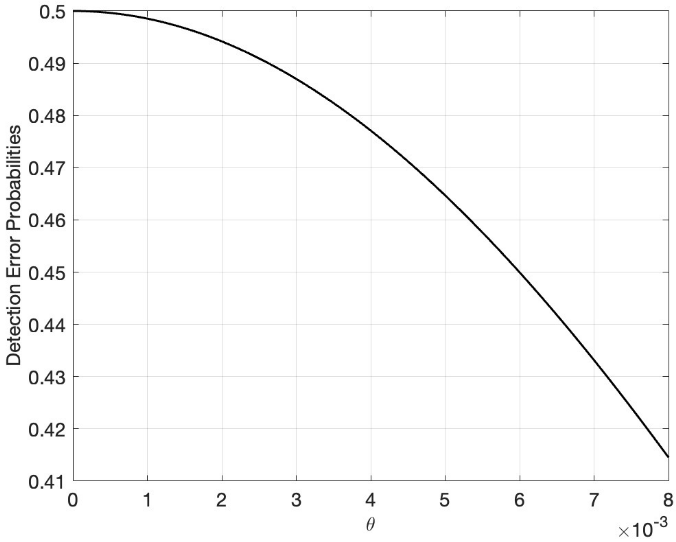


Figure 11: Detection Error Probabilities